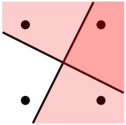
Capacity threshold for the Ising perceptron

Brice Huang (MIT)

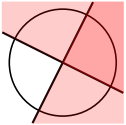
Brown Probability Seminar

April 29, 2025

The perceptron model



Ising



Spherical

Intersection of $\Sigma_N = \{-1,1\}^N$ or $\sqrt{N}\mathbb{S}^{N-1}$
with M i.i.d. random half-spaces

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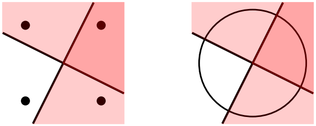

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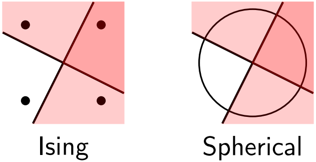

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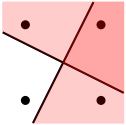
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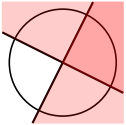
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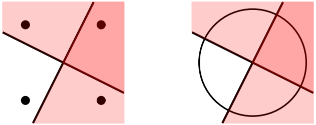

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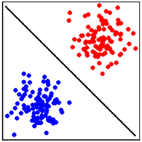
Application: linear classification

Given labeled dataset $(\boldsymbol{g}^1, y^1), \ldots, (\boldsymbol{g}^M, y^M) \in \mathbb{R}^N \times \{-1, 1\}$, is there a separating hyperplane?



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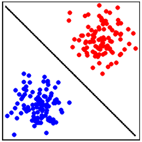

That is, does there exist $\boldsymbol{x}$ such that

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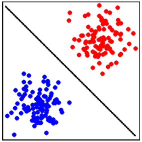

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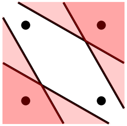
Consider **random labels** model:

$$\boldsymbol{g}^a \sim \mathcal{N}(0, I_N) \quad \text{independent of} \quad y^a \sim \mathrm{unif}(\pm 1).$$

This is equivalent to capacity problem! Hyperplane exists $\Leftrightarrow M/N < \alpha_*$.

Application: discrepancy minimization

Related model: **symmetric perceptron** with constraints $(\boldsymbol{g}^a, \boldsymbol{x})/\sqrt{N} \in [-\kappa, \kappa]$



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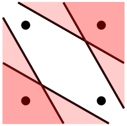

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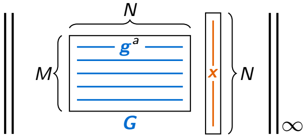
$\|\boldsymbol{G}\boldsymbol{x}\|_\infty =$ 

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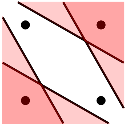

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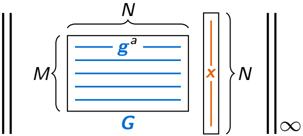

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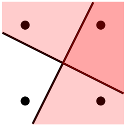
Applications: randomized control trials, sparsification, differential privacy, ...

Problem restatement: Ising perceptron



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Main result

Conjecture (Krauth Mézard 1989)

For the Ising perceptron, $\alpha_* = \alpha_{\mathsf{KM}} \approx 0.833$.

Theorem (Ding Sun 2018)

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Theorem (**H. 2024**)

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Both results hold for more general model with **margin** $\kappa \in \mathbb{R}$:

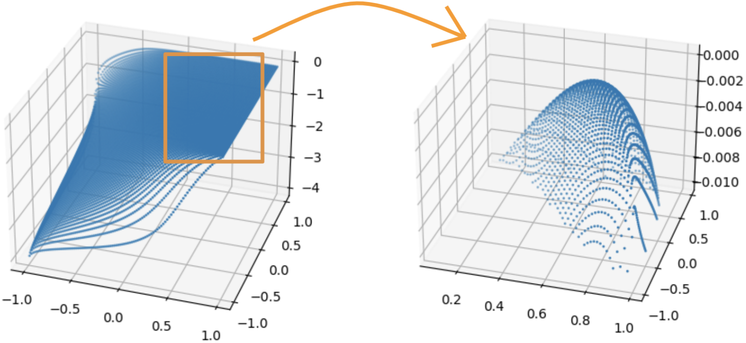
$$S = \left\{ \boldsymbol{x} \in \Sigma_N : \frac{(\boldsymbol{g}^a, \boldsymbol{x})}{\sqrt{N}} \geq \kappa, \quad \forall 1 \leq a \leq M \right\}$$

for suitable threshold $\alpha_{\mathsf{KM}}(\kappa)$, under further numerical conditions depending on κ.

The function $\mathscr{S}_*$ in our numerical condition

$\mathscr{S}_*(1,0) = 0$ local max, conjecturally unique global max

Plot of $\mathscr{S}_*$ (domain $\mathbb{R}^2$ reparametrized to $[-1,1]^2$):



$x, y \in [-1,1]$ $\qquad$ $\mathscr{S}_* \geq -0.01$

Background: physics predictions

Gardner Derrida 88, Krauth Mézard 89:

- Volume formula $\frac{1}{N} \log |S(\alpha N \,\text{constraints})| \xrightarrow{p} \mathrm{Vol}(\alpha)$ in terms of fixed pt eqn

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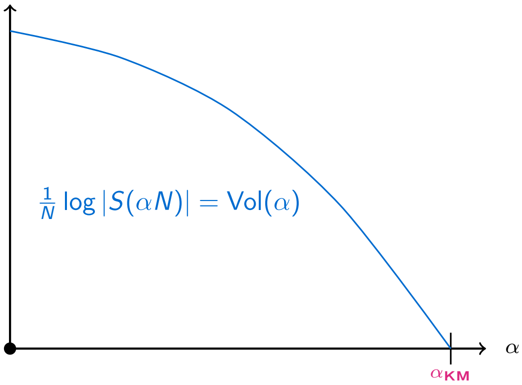
- Capacity: α_{KM} solves $\mathsf{Vol}(\alpha_{\mathsf{KM}}) = 0$



Rigorous results: positive spherical perceptron

For constraints $(\boldsymbol{g}^a, \boldsymbol{x})/\sqrt{N} \geq \kappa$, where $\kappa \geq 0$:

- Shcherbina Tirozzi 03: proof of volume limit $\mathrm{Vol}_\kappa(\alpha)$ (and thus capacity)
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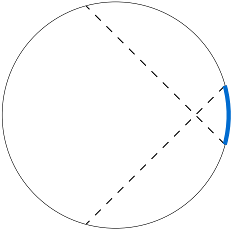
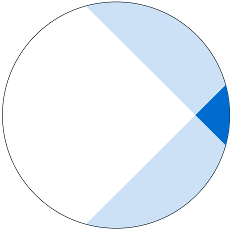
Crucial to proofs: $\kappa \geq 0$ spherical perceptron is **convex** problem



intersection nonempty

$\Longleftrightarrow$



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- $\mathbb{E}|S(N\alpha)| \ll 1 \Rightarrow$ no solution at constriant density α (whp)
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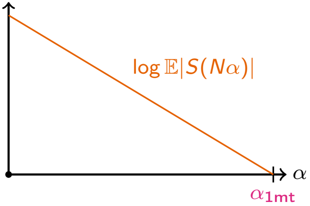

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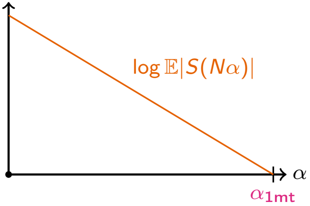
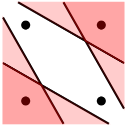


- (Hope to) show $\mathbb{E}[|S(N\alpha_{\mathsf{1mt}})|^2] \asymp (\mathbb{E}|S(N\alpha_{\mathsf{1mt}})|)^2 = 1$. If so, $\alpha_* = \alpha_{\mathsf{1mt}}$

1st/2nd moment method: a success story

This locates α_* in **symmetric Ising perceptron** with constraints $\frac{|\langle \boldsymbol{g}^a, \boldsymbol{x} \rangle|}{\sqrt{N}} \le \kappa$ (Aubin Perkins Zdeborová 19, Perkins Xu 21, Abbe Li Sly 22, ...)



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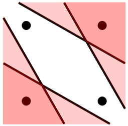

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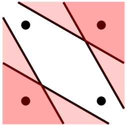

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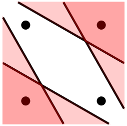

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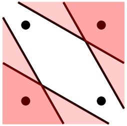

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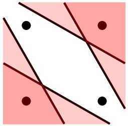

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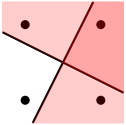
$\mathbb{E}|S(M)|^2$ can be calculated similarly, and moment method works.

Direct moment method fails in asymmetric model

In our model, $S(\alpha N) = \{\boldsymbol{x} \in \{\pm 1\}^N : \ (\boldsymbol{g}^a, \boldsymbol{x}) \geq 0 \quad \forall 1 \leq a \leq \alpha N\}$



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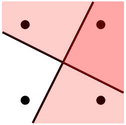

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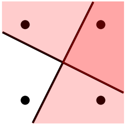

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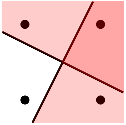

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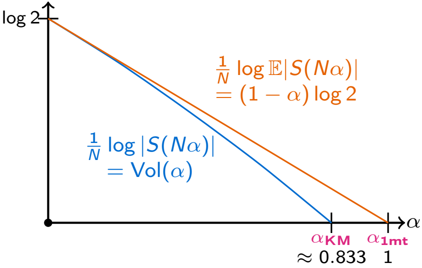

What goes wrong? Asymmetric vs. symmetric models

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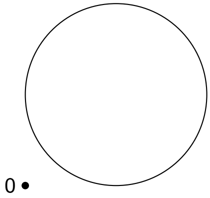

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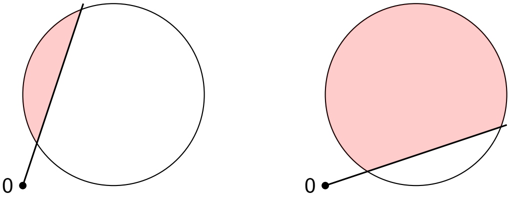

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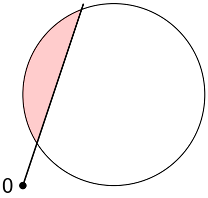

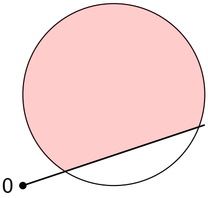

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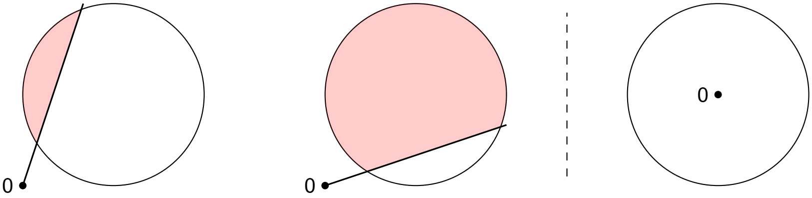

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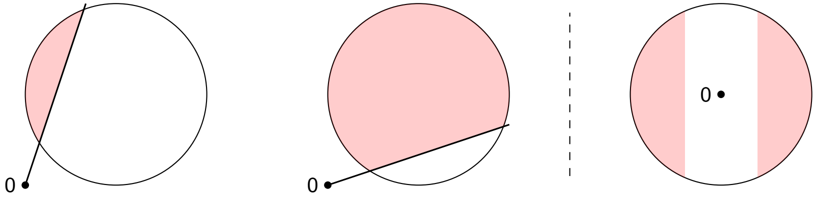

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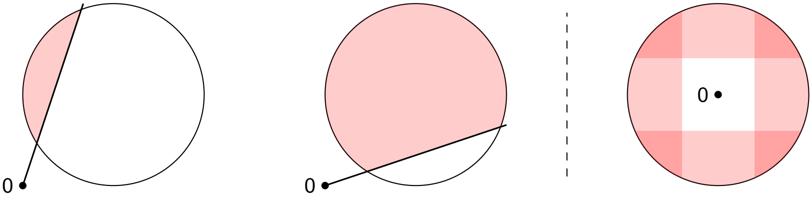

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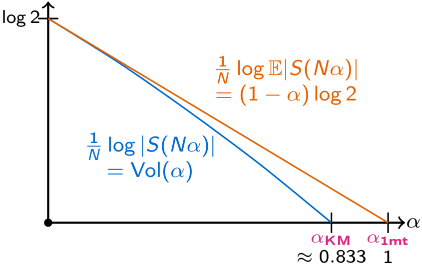
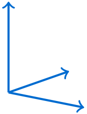
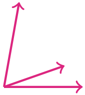
What goes wrong? A large deviations perspective



$\mathbb{E}|S(N\alpha)|$ dominated by events where the $\boldsymbol{g}^a$ are **atypically correlated**



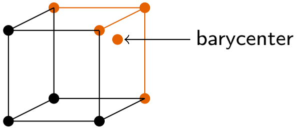
Typically: $\boldsymbol{g}^a$ orthogonal



Atypically: $\boldsymbol{g}^a$ correlated, which inflates # solutions

Conditional moment method

Key intuition of Ding Sun 18, Bolthausen 19: 1st mt failure caused by large deviation events in **barycenter** of solution set S



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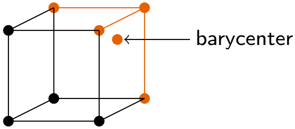

That is, $\mathbb{E}(|S|) \gg (\text{typical } |S|)$ but we expect, for **typical** realization of barycenter:

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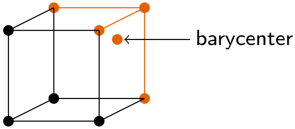

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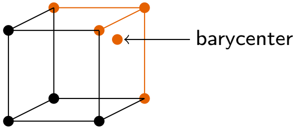

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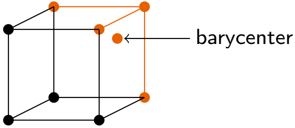

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Barycenter is mathematically complicated, but can condition on **physics proxy**

(Heuristic) physics description of barycenter

TAP equation: nonlinear system in

- $\boldsymbol{G} \in \mathbb{R}^{M \times N}$ matrix with rows $\boldsymbol{g}^1, \ldots, \boldsymbol{g}^M$
- $\boldsymbol{m} \in \mathbb{R}^N$ barycenter of S
- $\boldsymbol{n} \in \mathbb{R}^M$ average slacks of constraints: $n_a = \mathrm{avg}_{\boldsymbol{x} \in S}\left\{\frac{\langle \boldsymbol{g}^a, \boldsymbol{x}\rangle}{\sqrt{N}}\right\}$

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For explicit nonlinearities $\dot{F}, \widehat{F} : \mathbb{R} \to \mathbb{R}$, constants b, d:

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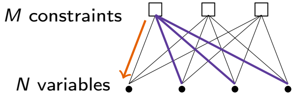
($\leftrightarrow$ dense graph limit of **belief propagation**)



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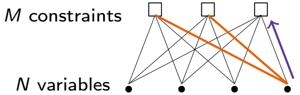

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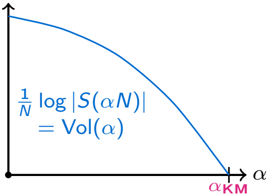

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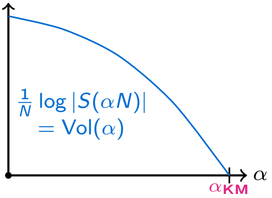

Proof roadmap

True model:
$\boldsymbol{G}$ iid gaussian

Conjecturally similar
under TAP heuristic

Planted model:
$\boldsymbol{G}$ cond on
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1st/2nd mt method (direct calculation, numerical conditions enter here)

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[Previous work: **motivation** only]

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DS18: lower bd by 1st/2nd mt on **truncation** of $|S|$

1st/2nd mt method (direct calculation, numerical conditions enter here)

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Follows if TAP heuristic proven

$\alpha_* = \alpha_{\mathsf{KM}}$ in planted model

Proof roadmap

True model:
G iid gaussian

Conjecturally similar
under TAP heuristic

Our work proves this...

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...justifying this implication

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$\longleftrightarrow$

We now explain how to make this step rigorous.

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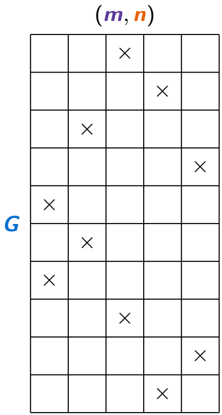
Key issue: linking true and planted models



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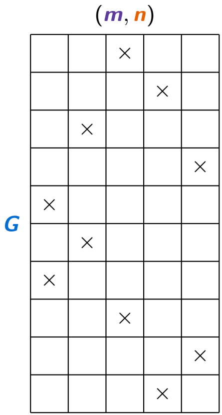

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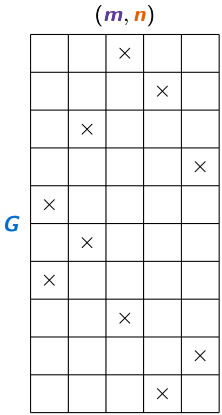

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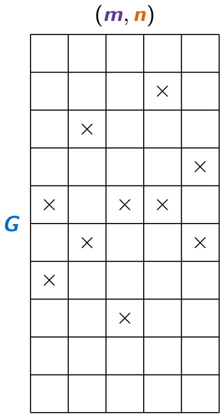

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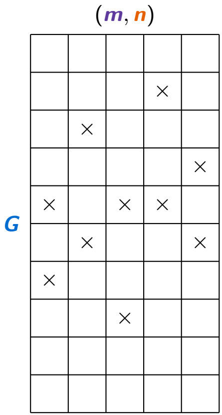

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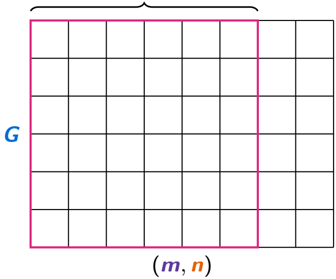
$\Rightarrow$ planted / true models can a priori be very different

This work: contiguity of true / planted models

$T = \{$"typical" $(m, n)\}$ (suitably defined set; whp in planted model)



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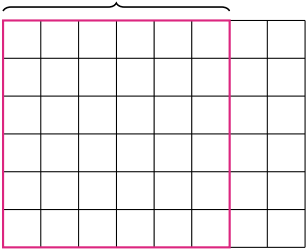

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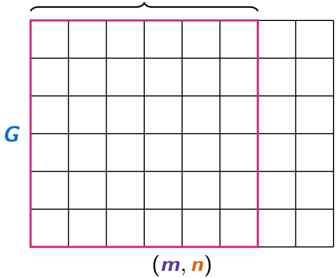

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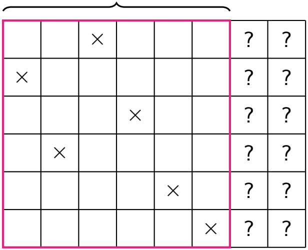

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This shows true $\approx$ planted. Formally, $\forall$ event E,

$$\mathbb{P}_{\text{true}}(E) \quad \le \quad O(1) \cdot \sup_{(\boldsymbol{m},\boldsymbol{n}) \in \boldsymbol{T}} \mathbb{P}_{\text{planted}}(E | \boldsymbol{m}, \boldsymbol{n}) + o(1)$$

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   ! This holds for k fixed as $N \to \infty$. Does not imply AMP finds **exact** crit pt

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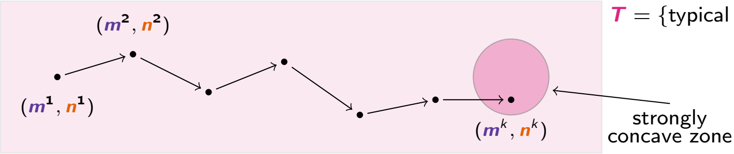

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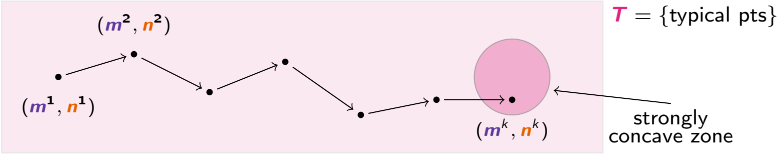

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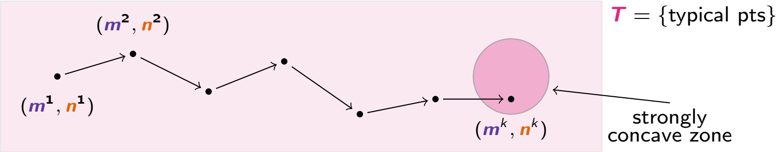

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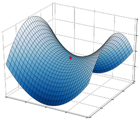

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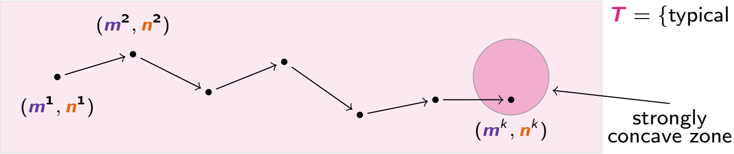

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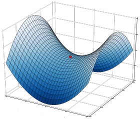

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Proof adapts AMP-conditioned gaussian comparison approach of Celentano 22

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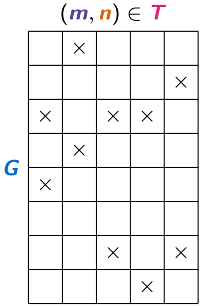

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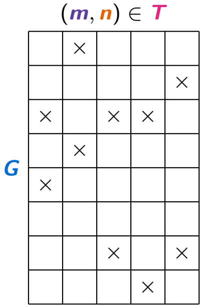

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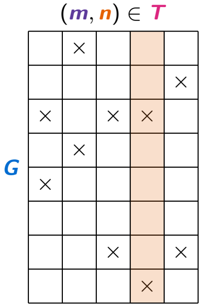

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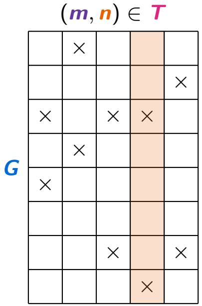

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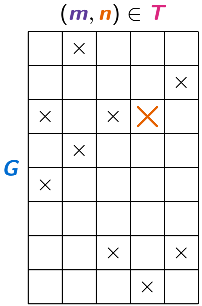

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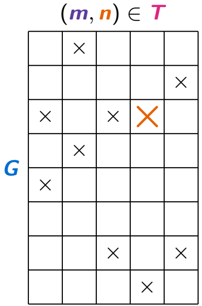

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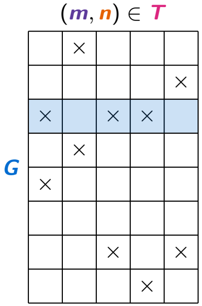

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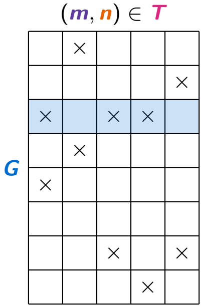

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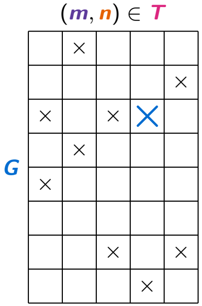

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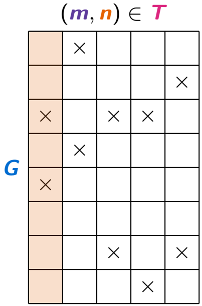

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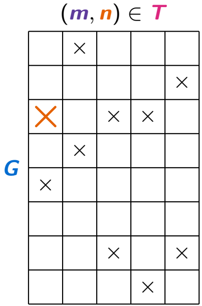

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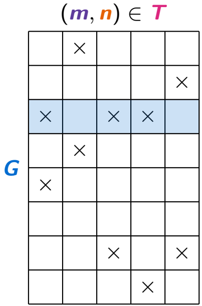

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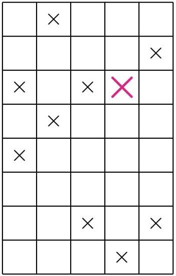

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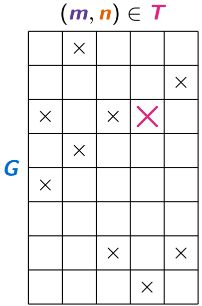

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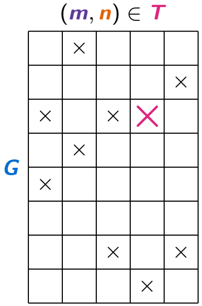

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If too many rows have > 1 $\times$s, claim cannot be true!

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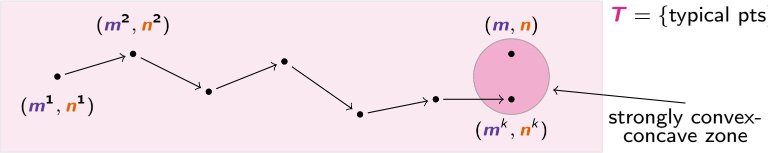

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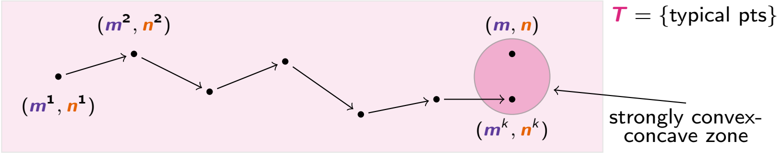

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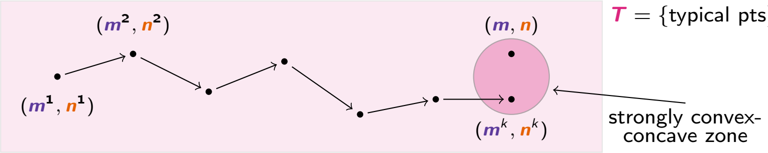

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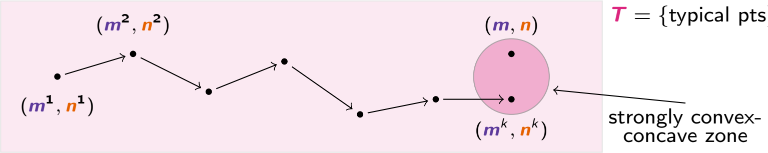

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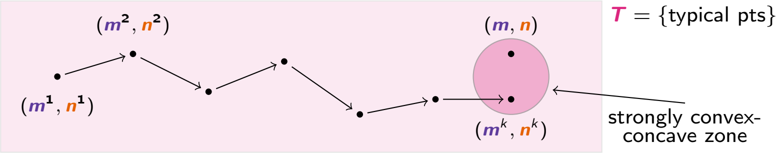

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- $\|(\boldsymbol{m}^k, \boldsymbol{n}^k) - (\boldsymbol{m}, \boldsymbol{n})\| = o_k(1)$

Recap: contiguity of true / planted models

$\boldsymbol{T} = \{\text{typical } (\boldsymbol{m}, \boldsymbol{n})\}$

			$\times$				?	?
	$\times$						?	?
				$\times$			?	?
$\boldsymbol{G}$		$\times$					?	?
					$\times$		?	?
						$\times$	?	?

$(\boldsymbol{m}, \boldsymbol{n})$

We show, for $\boldsymbol{G} \sim$ true model:

1. Existence: $\boldsymbol{G}$ has TAP solution $(\boldsymbol{m}, \boldsymbol{n}) \in \boldsymbol{T}$ whp (most rows have a $\times$)
2. Uniqueness: $\mathbb{E}[\#\text{TAP solutions in } \boldsymbol{T}] = 1 + o(1)$ (rows average $1 + o(1)$ $\times$'s)

This shows true $\approx$ planted.

Recap: proof roadmap

True model:
$\boldsymbol{G}$ iid gaussian

Conjecturally similar
under TAP heuristic

Our work proves this. . .

Planted model:
$\boldsymbol{G}$ cond on
TAP($\boldsymbol{G}, \boldsymbol{m}, \boldsymbol{n}$)

1st/2nd mt
method

$\alpha_* = \alpha_{\mathsf{KM}}$ in
true model

. . .**justifying this implication**

Follows if TAP heuristic proven

$\alpha_* = \alpha_{\mathsf{KM}}$ in
planted model

Conclusion

- We show contiguity of true model & model with planted TAP fixed point
- Then, 1st/2nd moment in planted model locates capacity $\alpha_* = \alpha_{\mathsf{KM}}$

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- **Open Q**: capacity of non "replica symmetric" models? E.g. spherical $\kappa < 0$

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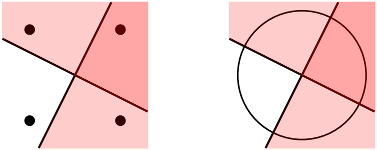

Thank you!

Choice of typical set T

Define the **coordinate profiles**

$$\mu(\boldsymbol{m}) = \frac{1}{N}\sum_{i=1}^{N}\delta(m_i) \qquad \nu(\boldsymbol{n}) = \frac{1}{M}\sum_{a=1}^{M}\delta(n_a) \qquad \in \mathcal{P}(\mathbb{R})$$

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Choice: $\boldsymbol{T} = \{\, (\boldsymbol{m}, \boldsymbol{n}) \;:\; \mu(\boldsymbol{m}) \approx \mu_*,\ \nu(\boldsymbol{n}) \approx \nu_* \,\}$

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Arises in proof steps:

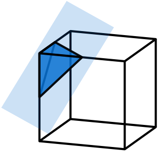
- For $(\boldsymbol{m}, \boldsymbol{n}) \in \boldsymbol{T}$, 1st/2nd mt method works conditional on $(\boldsymbol{m}, \boldsymbol{n})$.
- AMP state evolution characterizes coordinate profile of iterates $\boldsymbol{m}^k, \boldsymbol{n}^k$. This allows us to show $(\boldsymbol{m}^k, \boldsymbol{n}^k) \in \boldsymbol{T}$.

Earlier work: AMP-conditioned moment method

$$\boldsymbol{m}^{k+1} = \dot{F}\left(\frac{\boldsymbol{G}^\top \boldsymbol{n}^k}{\sqrt{N}} - d\boldsymbol{m}^k\right) \qquad \boldsymbol{n}^k = \widehat{F}\left(\frac{\boldsymbol{G}\boldsymbol{m}^k}{\sqrt{N}} - b\boldsymbol{n}^{k-1}\right)$$

Plan: for large $k = O(1)$, condition on $\boldsymbol{m}^1, \boldsymbol{n}^1, \ldots, \boldsymbol{m}^k, \boldsymbol{n}^k$, then 1st/2nd moment

No existence / uniqueness issue, but now $\mathbb{E}[|S(M)|\,|\text{AMP}]$ is k-dim optimization
(Over codimension-k slices of $\{\pm 1\}^N$ orthogonal to $\boldsymbol{m}^1, \ldots, \boldsymbol{m}^k$)



DS18: **for lower bound**, tractable 1st/2nd moment on **truncated** count

$$|S(M) \cap \{\text{correct affine slice}\}|$$

Upper bound: can't do truncation, optimization intractable