Capacity threshold for the Ising perceptron

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MIT Combinatorics Seminar

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High-dimensional disordered systems

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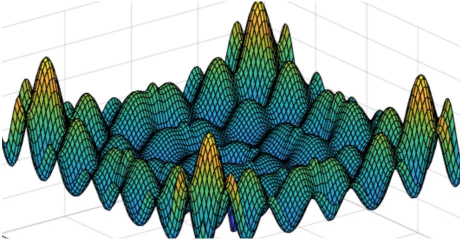

Highly non-convex landscapes, often with exponentially many maxima

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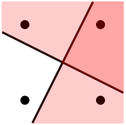
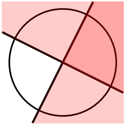
Goal: unified rigorous theory & broadly applicable tools

The perceptron model


Ising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Intersection of $\Sigma_N = \{-1,1\}^N$ or $\sqrt{N}\mathbb{S}^{N-1}$ with M i.i.d. random half-spaces

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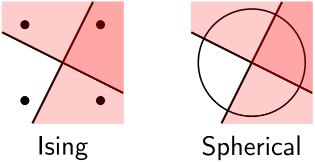

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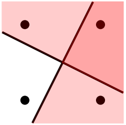

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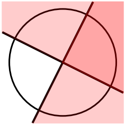

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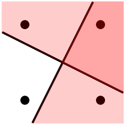

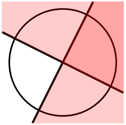

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Main question: what is $\alpha_* = \alpha_*(\kappa) = \text{p-lim}_{N\to\infty} M/N$?

Connections to other problems

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- $S(M) \leftrightarrow$ synaptic weights memorizing all M patterns
- Capacity $\leftrightarrow$ max # patterns neural network can memorize

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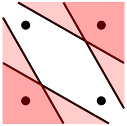
Related model: **symmetric perceptron** with constraints $\langle \boldsymbol{g}^a, \boldsymbol{x} \rangle / \sqrt{N} \in [-\kappa, \kappa]$



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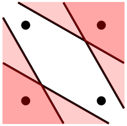

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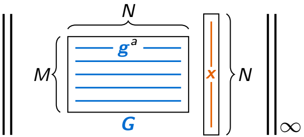
$\|\boldsymbol{G}\boldsymbol{x}\|_\infty =$ 

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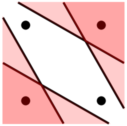

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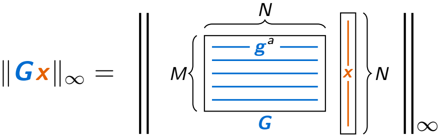

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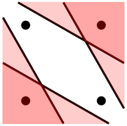

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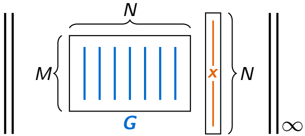
$\|\boldsymbol{G}\boldsymbol{x}\|_\infty =$ 

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For the $\kappa = 0$ Ising perceptron, $\alpha_* = \alpha_{\mathsf{KM}} \approx 0.833$.

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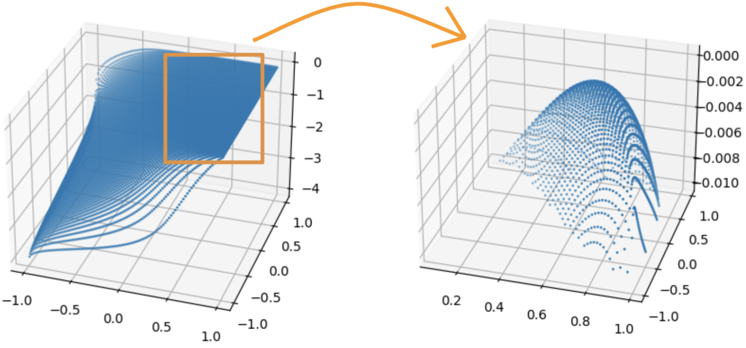
(Also for general $\kappa \in \mathbb{R}$, under several numerical conditions depending on κ)

The function $\mathscr{S}_*$ in our numerical condition

$\mathscr{S}_*(1, 0) = 0$ local max, conjecturally unique global max

Plot of $(x, y) \mapsto \mathscr{S}_*(\tanh^{-1}(x), \tanh^{-1}(y))$:



$x, y \in [-1, 1]$ $\qquad$ $\mathscr{S}_* \geq -0.01$

Background: physics predictions

Gardner Derrida 88, Krauth Mézard 89:

- Volume formula $\frac{1}{N} \log |S(\alpha N)| \to_p \mathrm{Vol}(\kappa, \alpha)$ in terms of fixed point eqn
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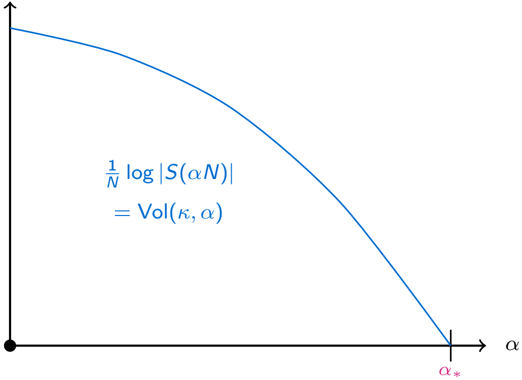
- Capacity: $\alpha_* = \alpha_*(\kappa)$ solves $\mathrm{Vol}(\kappa, \alpha_*) = 0$



Spherical perceptron, $\kappa \geq 0$

- Shcherbina Tirozzi 03: proof of volume limit $\mathrm{Vol}(\kappa, \alpha)$ (and thus capacity)
- Stojnic 13: simple proof of capacity threshold

Crucial to proofs: $\kappa \geq 0$ spherical perceptron is **convex** problem!

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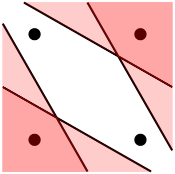

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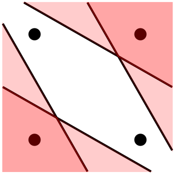

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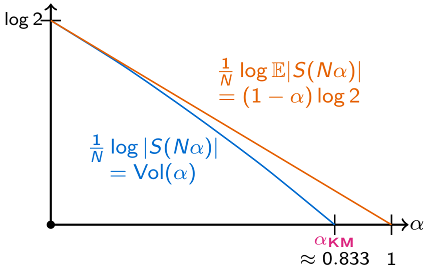
But fails in our model! $\mathbb{E}|S(N\alpha)| = 2^{(1-\alpha)N} \ll 1$ only when $\alpha > 1$

Failure of direct moment method



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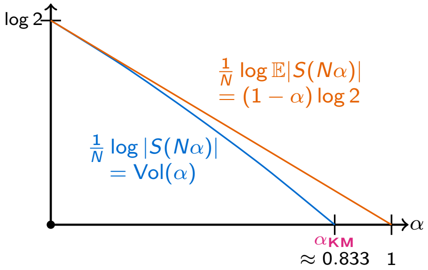
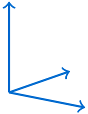
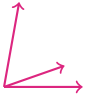


$\mathbb{E}|S(N\alpha)|$ dominated by events where the $\boldsymbol{g}^a$ are **atypically correlated**



Typically: $\boldsymbol{g}^a$ orthogonal



Atypically: $\boldsymbol{g}^a$ correlated, which inflates # solutions

Another perspective on 1st moment failure

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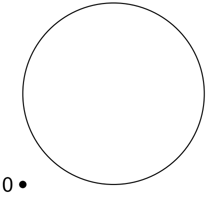

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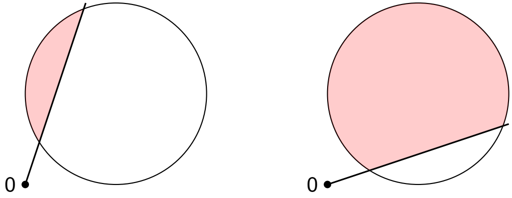

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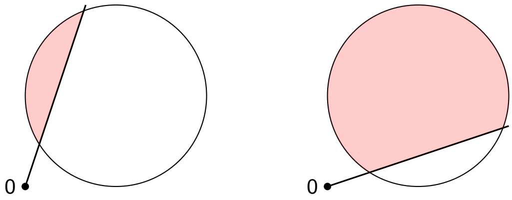

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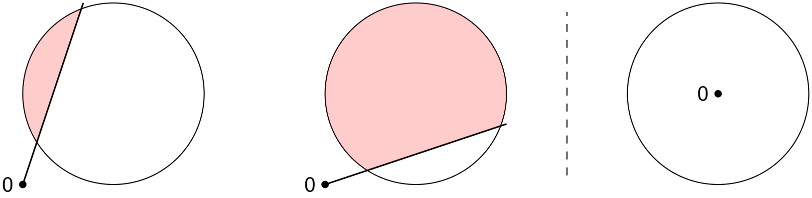

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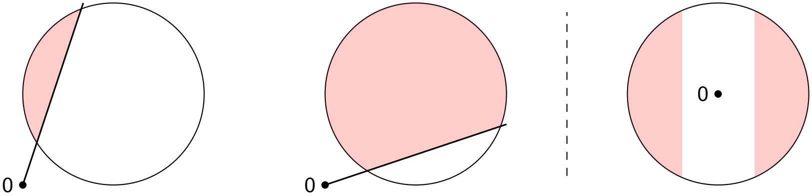

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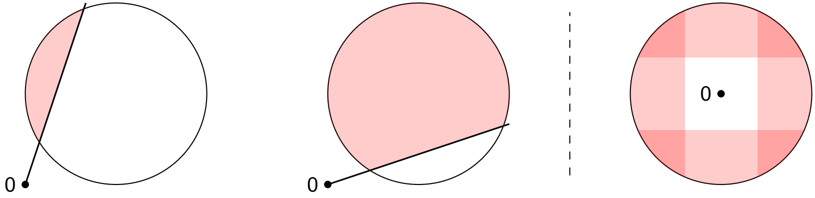

Each slice deletes half of solutions **on average**, but **genuine fluctuations**

Imagine simplified model: each slice deletes 1% or 99%. Then all solutions deleted much faster than 1st moment bound.

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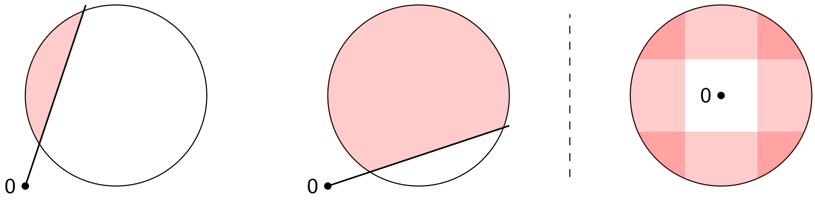

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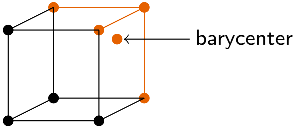
Next few slides: non-rigorous physics intuitions on how to remedy this.

Conditional moment method

Key intuition of Ding Sun 18, Bolthausen 19: 1st mt failure caused by large deviation events in **barycenter** of solution set $S(N\alpha)$



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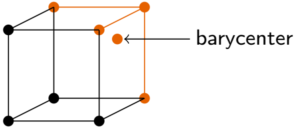

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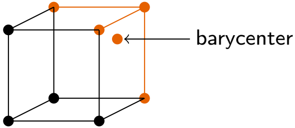

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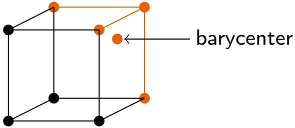

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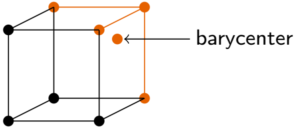

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This suggests plan: condition on **typical behavior** of barycenter, then 1st/2nd mt

Barycenter is mathematically complicated, but can condition on **physics proxy**

(Heuristic) physics description of barycenter

TAP equation: nonlinear system in

- $\boldsymbol{G} \in \mathbb{R}^{M \times N}$ matrix with rows $\boldsymbol{g}^1, \ldots, \boldsymbol{g}^M$
- $\boldsymbol{m} \in \mathbb{R}^N$ barycenter of $S(M)$
- $\boldsymbol{n} \in \mathbb{R}^M$ average slacks of constraints: $n_a = \text{avg}_{\boldsymbol{x} \in S(M)} \left\{ \frac{\langle \boldsymbol{g}^a, \boldsymbol{x} \rangle}{\sqrt{N}} - \kappa \right\}$

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For explicit nonlinearities $\dot{F}, \widehat{F}$, constants b, d:

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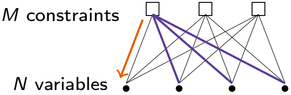
($\leftrightarrow$ dense graph limit of **belief propagation**)



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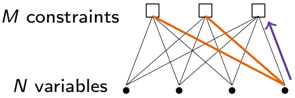

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Physics predictions for TAP & planted model

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1. Sample $(\boldsymbol{m}, \boldsymbol{n})$ from its law (explicit physics prediction)
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! existence/uniqueness of $(\boldsymbol{m}, \boldsymbol{n})$ is **not proven**, so planted $\neq$ true possible

Proof roadmap

True model:
G iid gaussian

Conjecturally similar
under TAP heuristic

$\leftarrow - - - - - - - - - - - \rightarrow$

Planted model:
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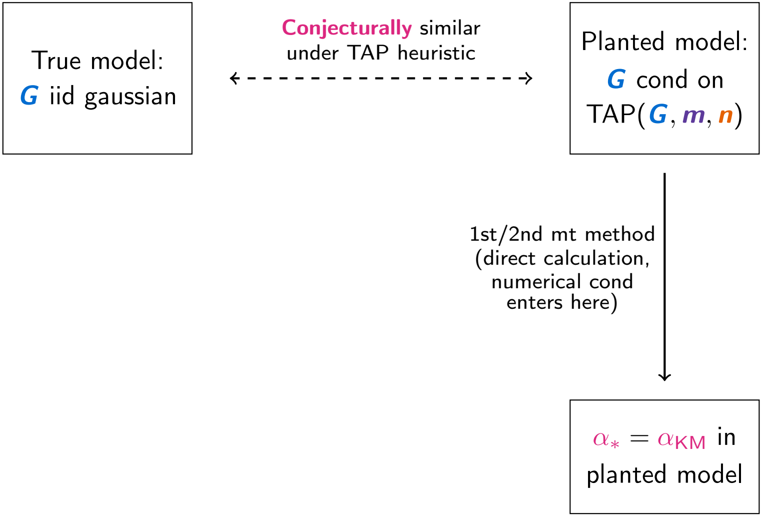
Proof roadmap



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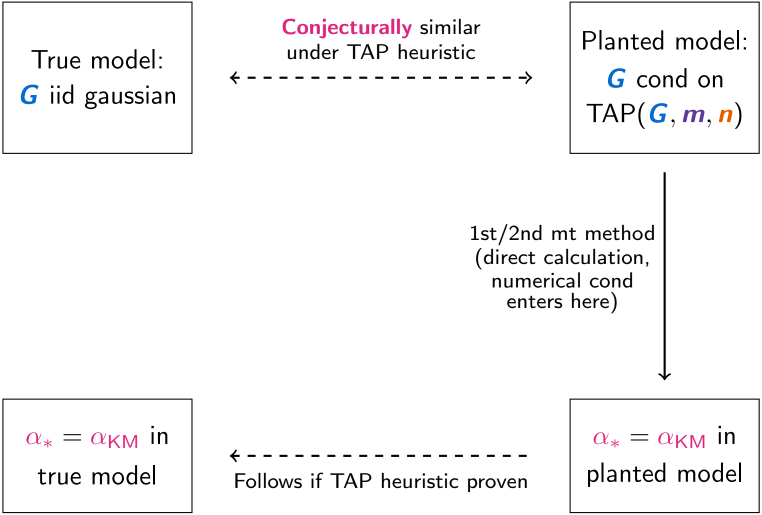

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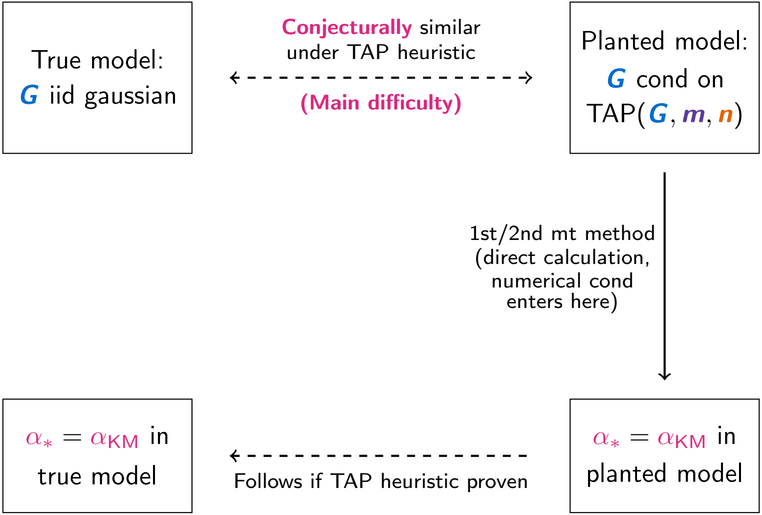

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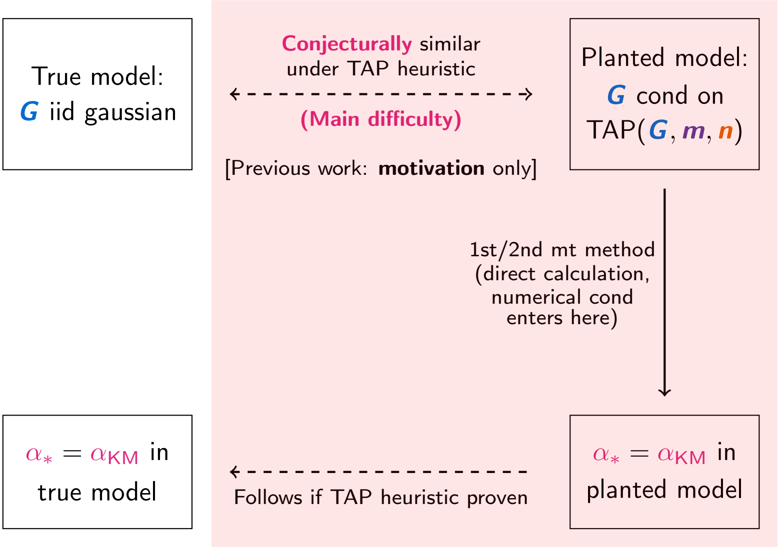

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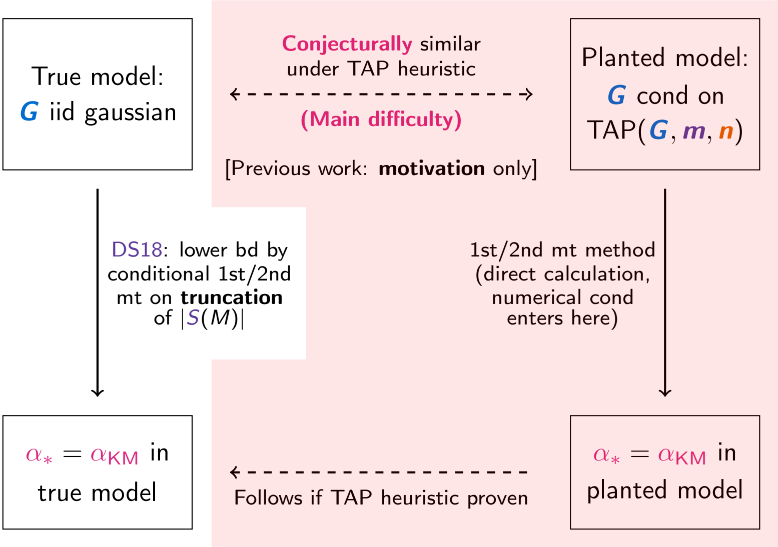

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Our work proves this...

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DS18: lower bd by conditional 1st/2nd mt on **truncation** of $|S(M)|$

1st/2nd mt method (direct calculation, numerical cond enters here)

$\alpha_* = \alpha_{\mathsf{KM}}$ in true model

Follows if TAP heuristic proven

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...justifying this implication

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Properties making the planted model tractable

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$$\mu(\boldsymbol{m}) = \frac{1}{N}\sum_{i=1}^{N} \delta(m_i) \qquad \nu(\boldsymbol{n}) = \frac{1}{M}\sum_{a=1}^{M} \delta(n_a) \qquad \in \mathcal{P}(\mathbb{R})$$

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2. $\mathrm{TAP}(\boldsymbol{G}, \boldsymbol{m}, \boldsymbol{n})$ is **linear** constraint on $\boldsymbol{G} \Rightarrow \boldsymbol{G}$ conditionally gaussian!

Moment calculation in planted model

Plan: 1st/2nd moment method on $|S| = \#(\text{solutions})$ **conditional on** $(\boldsymbol{m}, \boldsymbol{n})$

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$\boldsymbol{G}$ is **spiked gaussian mtx** $\Rightarrow$ $\boldsymbol{G}\boldsymbol{x}$ is **gaussian vector** whose law depends on only

$$a = \langle \boldsymbol{x}, \boldsymbol{m} \rangle \qquad b = \langle \boldsymbol{x}, \boldsymbol{H} \rangle \qquad\qquad (\boldsymbol{H} = \dot{F}^{-1}(\boldsymbol{m}))$$

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$\Rightarrow \mathbb{E}[|S| \,|\, \boldsymbol{m}, \boldsymbol{n}] \approx \exp(N \cdot \max f(a, b))$ essentially **2 variable maximization**

1st/2nd moment in planted model

1st/2nd moments are both explicit $O(1)$-variable maximizations.

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1st/2nd moment works!! Conditional on typical $(\boldsymbol{m}, \boldsymbol{n})$,

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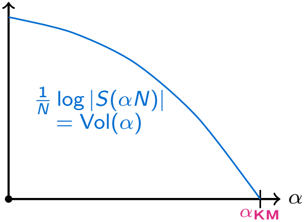
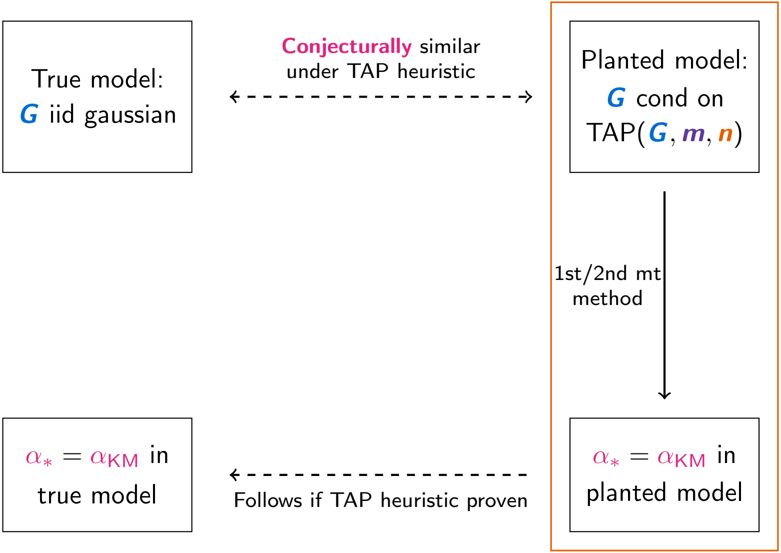


Since Vol has root α_{KM}, planted model has capacity α_{KM}

Proof roadmap



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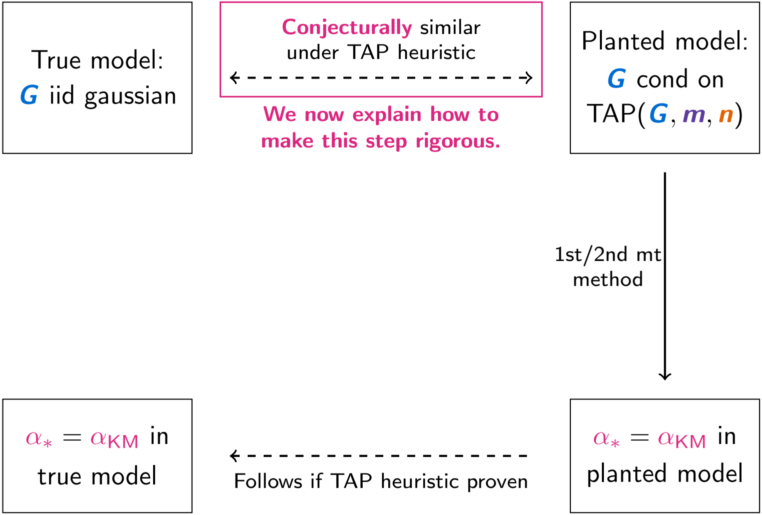
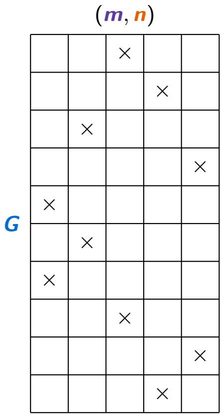


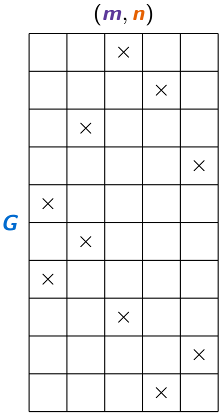
Key issue: linking true and planted models



True model $\leftrightarrow$ random row

Planted model $\leftrightarrow$ random col, then random $\times$ in col

Key issue: linking true and planted models



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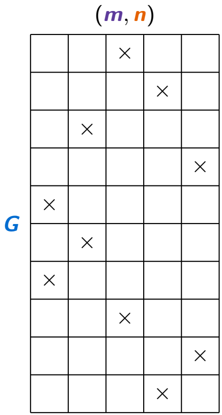

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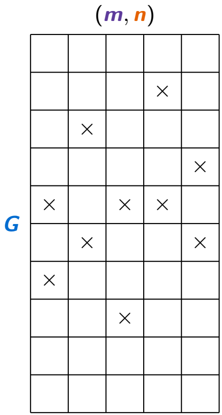

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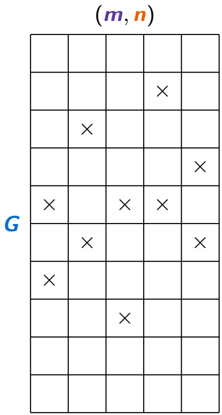

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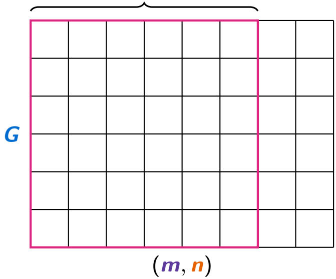
$\Rightarrow$ planted / true models can a priori be very different

This work: contiguity of true / planted models

$T = \{(m, n)$ with $\approx \underbrace{\text{ideal coordinate profiles}}\}$ (high-probability set)



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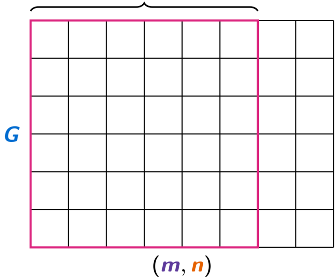

We show, for $\boldsymbol{G} \sim$ true model:

1. Existence: $\boldsymbol{G}$ has TAP solution $(\boldsymbol{m}, \boldsymbol{n}) \in \boldsymbol{T}$ whp (most rows have a $\times$)
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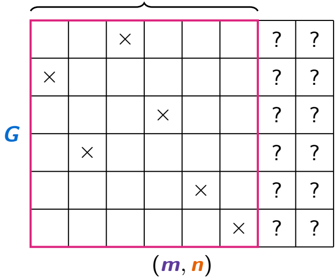

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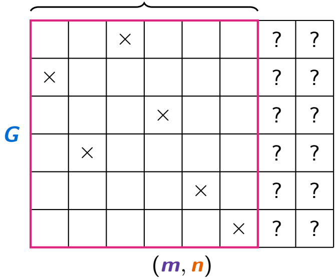

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This shows true $\approx$ planted. That is, $\forall$ event E,

$$\mathbb{P}_{\text{true}}(E) \quad \leq \quad C \sup_{(m,n) \in T} \mathbb{P}_{\text{planted}}(E | m, n) + o(1)$$

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That is, AMP finds an **approximate** critical point in $\boldsymbol{T}$

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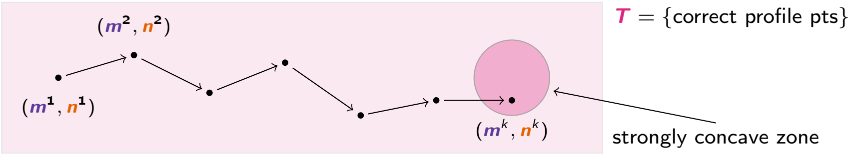

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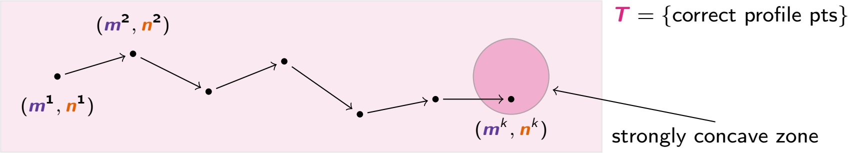

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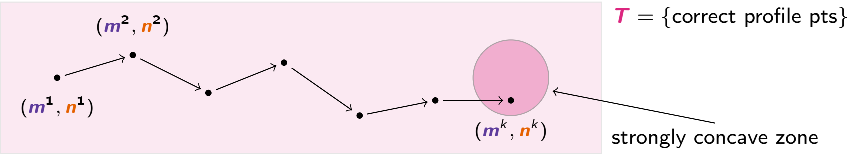

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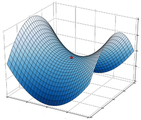

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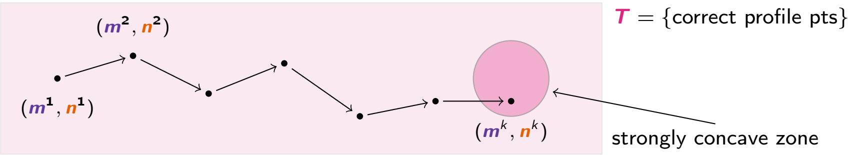

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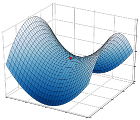

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Proof adapts AMP-conditioned gaussian comparison approach of Celentano 22

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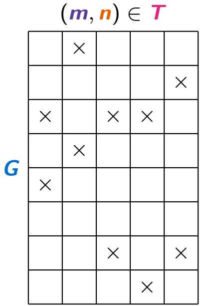

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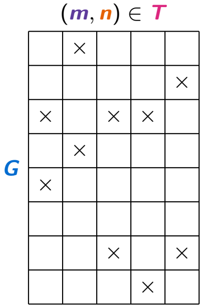

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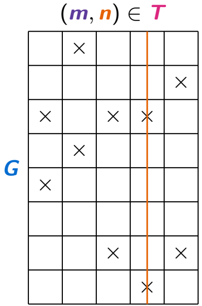

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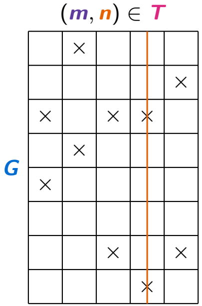

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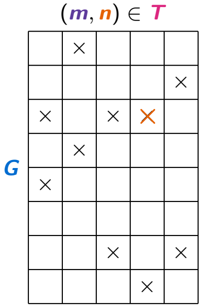

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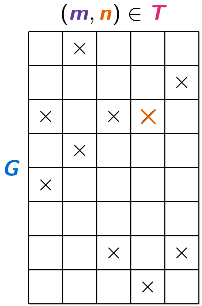

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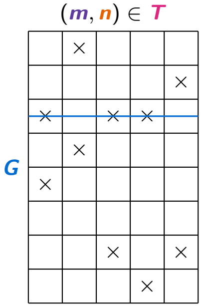

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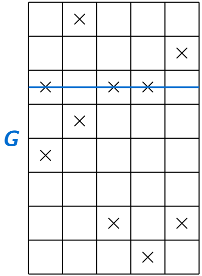

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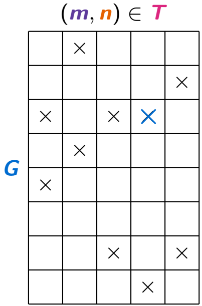

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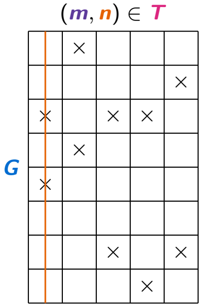

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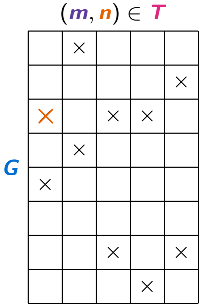

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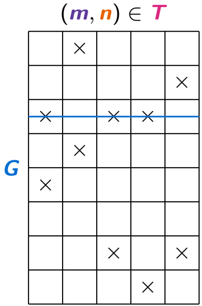

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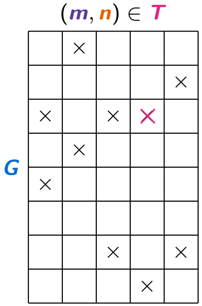

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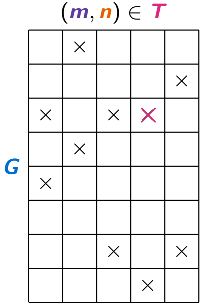

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Run AMP on disorder G

Did AMP return to (m, n)?

Experiment succeeds for at most one $\times$ per row

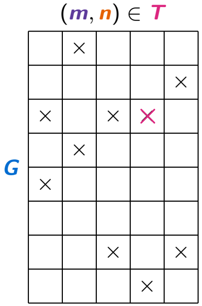
Uniqueness: double-counting argument

Want: for $\boldsymbol{G} \sim$ true model, $\mathbb{E}[\#\text{TAP fixed pts of } \boldsymbol{G} \text{ in } \boldsymbol{T}] = 1 + o(1)$

This also has an algorithmic proof! Following claim implies uniqueness:

Fix $(\boldsymbol{m}, \boldsymbol{n}) \in \boldsymbol{T} = \{\text{correct profiles}\}$. Sample $\boldsymbol{G}$ conditioned on $\text{TAP}(\boldsymbol{G}, \boldsymbol{m}, \boldsymbol{n})$.

AMP run on $\boldsymbol{G}$ finds **the planted point** $(\boldsymbol{m}, \boldsymbol{n})$ whp



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Run AMP on disorder $\boldsymbol{G}$

Did AMP return to $(\boldsymbol{m}, \boldsymbol{n})$?

Experiment succeeds for at most one $\times$ per row

If too many rows have > 1 $\times$s, claim cannot be true!

Uniqueness: AMP returns home in planted model

Want: for $(\boldsymbol{m}, \boldsymbol{n}) \in \boldsymbol{T}$, $\boldsymbol{G}$ conditioned on $\text{TAP}(\boldsymbol{G}, \boldsymbol{m}, \boldsymbol{n})$,

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Uniqueness: AMP returns home in planted model

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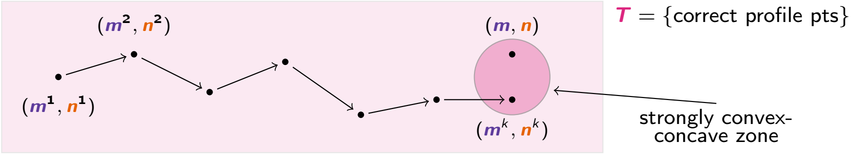
This can be proved by the same strategy!



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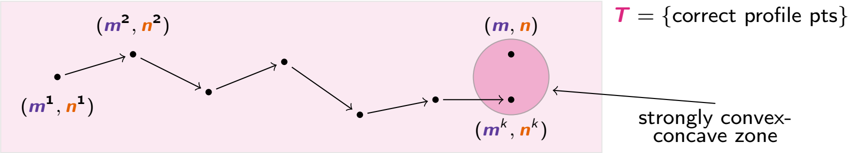

By adapting state evolution & gaussian comparison analyses to planted model:

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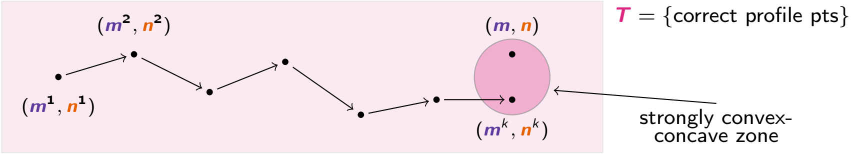

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- $(\boldsymbol{m}^k, \boldsymbol{n}^k) \in \boldsymbol{T}$, $\|\nabla \mathcal{F}_{\mathrm{TAP}}(\boldsymbol{m}^k, \boldsymbol{n}^k)\| = o_k(1)$, and $\|(\boldsymbol{m}^k, \boldsymbol{n}^k) - (\boldsymbol{m}, \boldsymbol{n})\| = o_k(1)$

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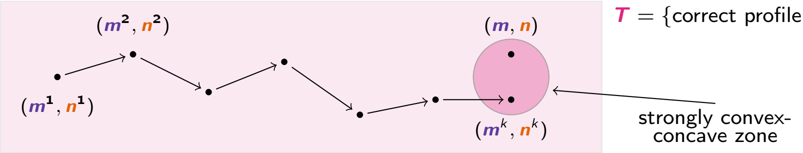

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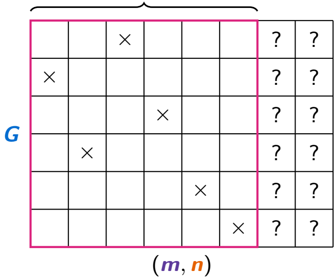
- $\mathcal{F}_{\mathrm{TAP}}$ strongly convex-concave near $(\boldsymbol{m}^k, \boldsymbol{n}^k)$

Recap: contiguity of true / planted models

$T = \{(m, n) \text{ with} \approx \text{ideal coordinate profiles}\}$



We show, for $G \sim$ true model:

1. Existence: G has TAP solution $(m, n) \in T$ whp (most rows have a $\times$)
2. Uniqueness: $\mathbb{E}[\#\text{TAP solutions in } T] = 1 + o(1)$ (rows average $1 + o(1)$ $\times$'s)

This shows true $\approx$ planted.

Recap: proof roadmap

True model:
G iid gaussian

Conjecturally similar
under TAP heuristic

Our work proves this. . .

Planted model:
G cond on
TAP(**G**, **m**, **n**)

1st/2nd mt
method

$\alpha_* = \alpha_{\mathrm{KM}}$ in
true model

. . .justifying this implication

Follows if TAP heuristic proven

$\alpha_* = \alpha_{\mathrm{KM}}$ in
planted model

Conclusion

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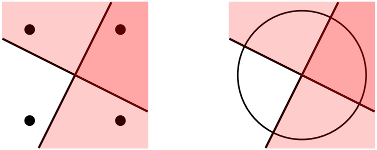


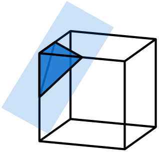
Thanks for your attention!

Earlier work: AMP-conditioned moment method

$$\boldsymbol{m}^{k+1} = \dot{F}\left(\frac{\boldsymbol{G}^\top \boldsymbol{n}^k}{\sqrt{N}} - d\boldsymbol{m}^k\right) \qquad \boldsymbol{n}^k = \widehat{F}\left(\frac{\boldsymbol{G}\boldsymbol{m}^k}{\sqrt{N}} - b\boldsymbol{n}^{k-1}\right)$$

Plan: for large $k = O(1)$, condition on $\boldsymbol{m}^1, \boldsymbol{n}^1, \ldots, \boldsymbol{m}^k, \boldsymbol{n}^k$, then 1st/2nd moment

No existence / uniqueness issue, but now $\mathbb{E}[|S(M)|\,|\text{AMP}]$ is k-dim optimization
(Over codimension-k slices of $\{\pm 1\}^N$ orthogonal to $\boldsymbol{m}^1, \ldots, \boldsymbol{m}^k$)



DS18: **for lower bound**, tractable 1st/2nd moment on **truncated** count

$$|S(M) \cap \{\text{correct affine slice}\}|$$

Upper bound: can't do truncation, optimization intractable