The Algorithmic Phase Transition of Random k-SAT for Low Degree Polynomials

Brice Huang (MIT)
Joint work with Guy Bresler

CanaDAM 2023

Random k-SAT

k-SAT formula: AND of m **clauses**, each an OR of k **literals**, e.g.

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Parameters: fix large k, $\alpha = \alpha(k)$. Then $m, n \to \infty$ with $m/n \to \alpha$

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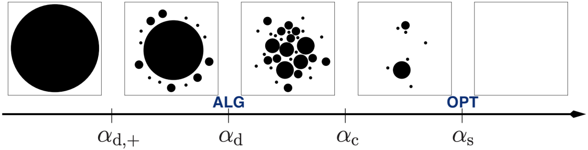
Heuristic: **shattering** at $\approx 2^k \log k/k$ obstructs algorithms (Achlioptas-Coja-Oghlan 08)



(Adapted from Krzakala-Montanari-Ricci-Tersenghi-Semerjian-Zdeborová 07)

Main Result (informal)

Theorem (Bresler-H. 21)

Low degree polynomial algorithms cannot succeed above $\alpha = 4.911 \cdot 2^k \log k / k$.

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Clause Density	Algorithm(s)	Reference
$C2^k/k$	DPLL algorithms	[Achlioptas-Beame-Molloy 04]
$(1+\varepsilon_k)2^k \log k/k$	Survey Propagation guided decimation	[Hetterich 16]
$(1+\varepsilon_k)2^{k-1} \log^2 k/k$	Balanced sequential local algorithms (NAE-k-SAT)	[Gamarnik-Sudan 17]
$C2^k \log^2 k/k$	Walksat	[Coja-Oghlan-Haqshenas-Hetterich 17]
$4.911 \cdot 2^k \log k/k$	Low degree polynomials	This work

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- i.e. $\varepsilon_k n$ Hamming distance to assignment satisfying $(1 - \varepsilon_k)m$ clauses

Main Result

$$\kappa^* = \min_{\beta > 1} \frac{\beta}{1 - \beta e^{-(\beta-1)}} \approx 4.911$$

Theorem (Bresler-H. 21)

If $\alpha > \kappa^* 2^k \log k/k$, *then no degree* $D = o(n/\log n)$ *polynomial succeeds with probability* $1 - \exp(-\Omega(D \log n))$.

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Theorem (Bresler-H. 21)

For the same α, no $O(1)$-local algorithm succeeds with probability $\exp(-O(n^{1/3}))$.

$O(1)$-local algorithms also simulate FIX.

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- Max independent set (Gamarnik-Sudan 14, Rahman-Virág 17, Gamarnik-Jagannath-Wein 20, Wein 20)
- Random NAE-k-SAT (Gamarnik-Sudan 17)
- Hypergraph maxcut (Chen-Gamarnik-Panchenko-Rahman 19)
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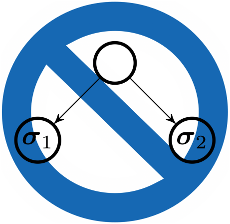
Overlap gap: no solutions x, y have **medium** Hamming distance $\in [\nu_1 n, \nu_2 n]$

- Intuition: solutions close together or far apart

Classic OGP (Gamarnik-Sudan 14, Gamarnik-Jagannath-Wein 20)

1. **Forbidden structure**: no solution pair medium distance apart



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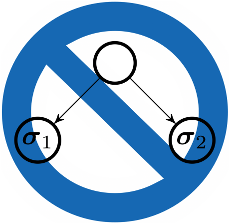

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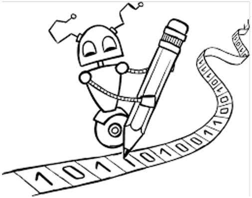

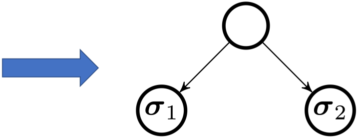

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Forbidden Structure: two assignments $y^{(1)}, y^{(2)} \in \{\mathtt{T}, \mathtt{F}\}^n$ such that

- Each $y^{(i)}$ satisfies some $\Phi^{(t_i)}$ (*not necessarily same* t_i)
- $d_H(y^{(1)}, y^{(2)}) \in [\nu_1 n, \nu_2 n]$

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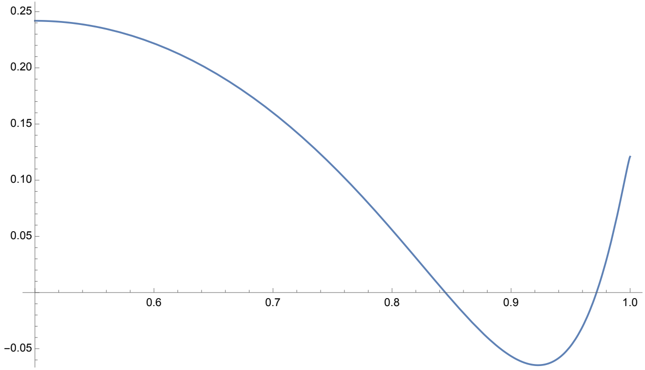
This structure doesn't occur w.h.p. by 1st moment calculation

$$\frac{1}{n} \log \mathbb{E} \# \left(\text{pairs of solutions } x, y \text{ with } d_H(x,y) \approx tn\right)$$



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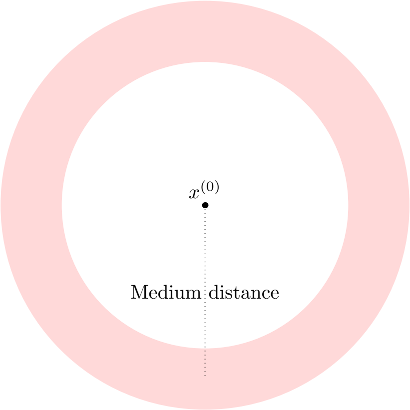

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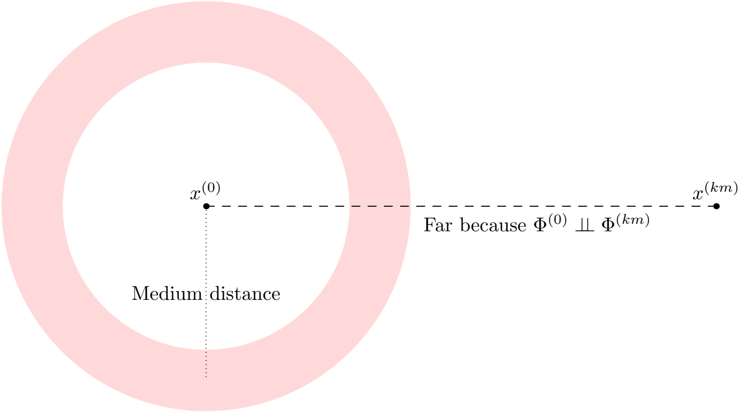

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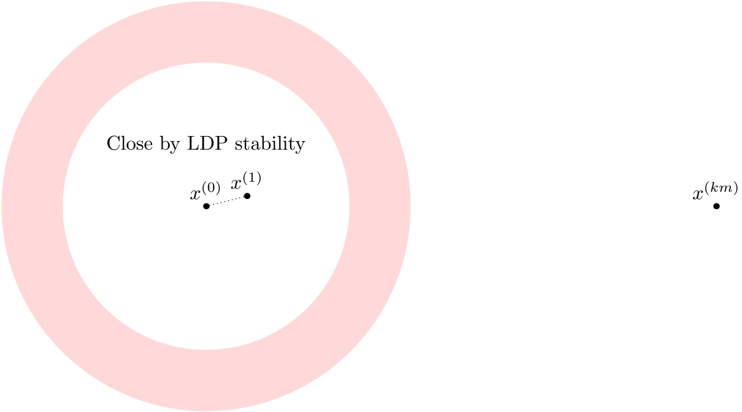

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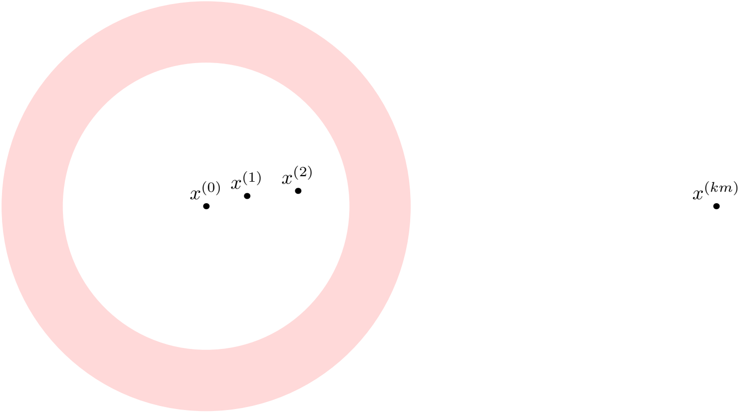

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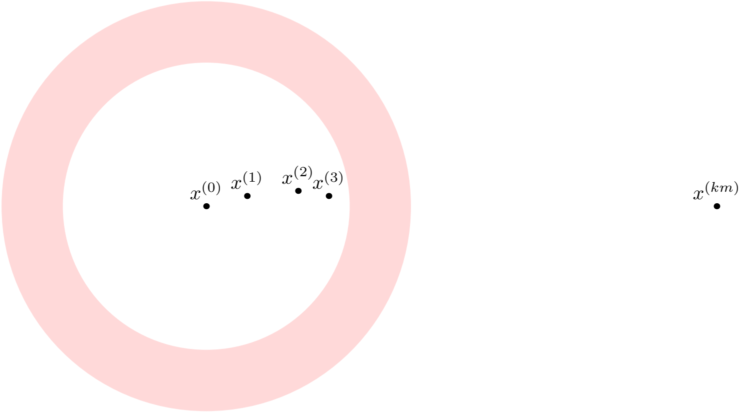

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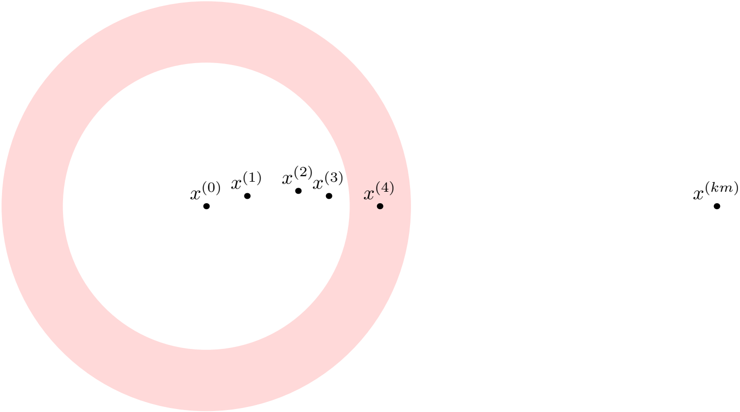

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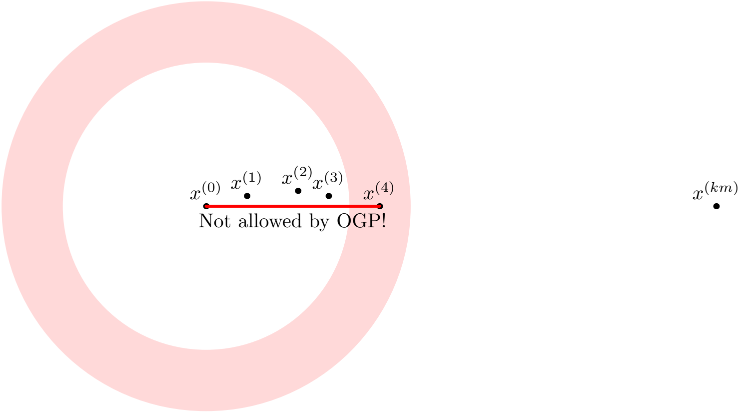
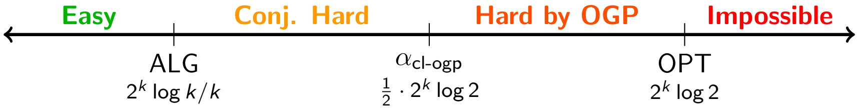


Contradiction $\Rightarrow$ $\mathcal{A}$ cannot succeed.

Classic OGP to Multi-OGP



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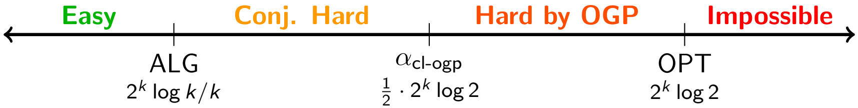

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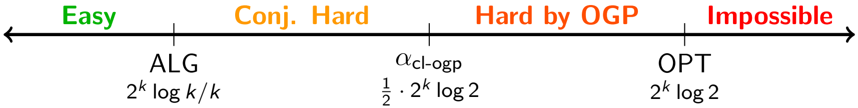

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Multi-OGP determined ALG for maximum independent set (Rahman-Virág 17, Wein 20) and mean-field spin glasses (H.-Sellke 21 & 23)

“Ladder” Multi-OGP (Wein 20)

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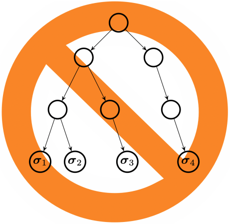

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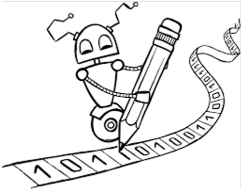

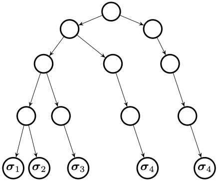

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Doesn't occur w.h.p. by 1st moment argument

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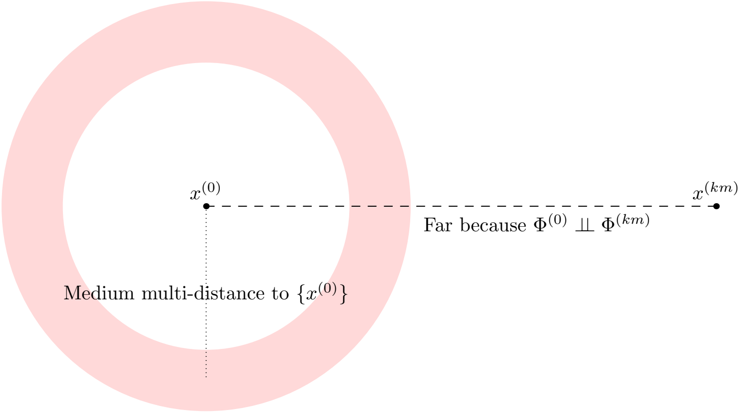

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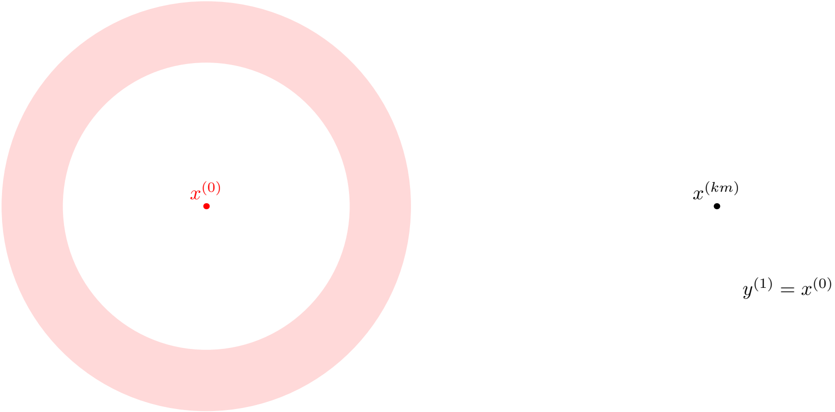

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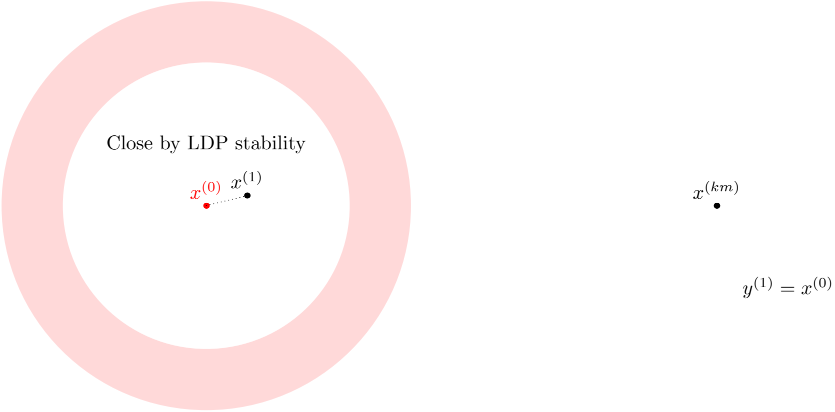

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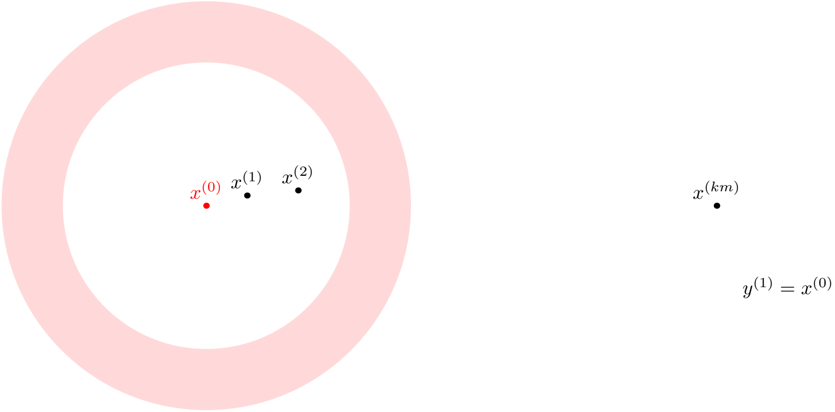

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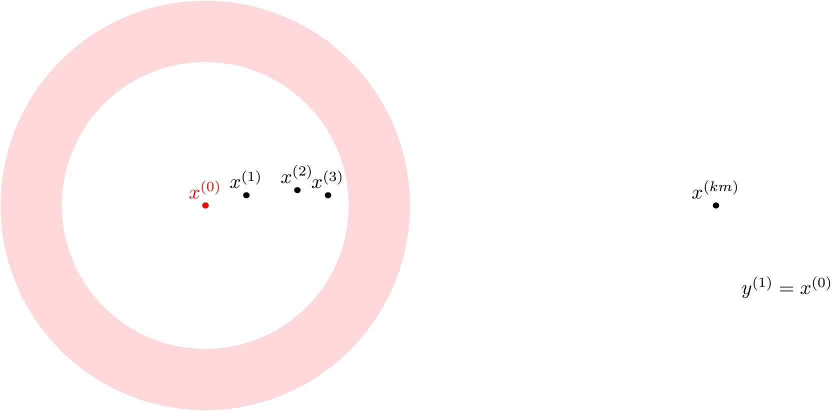

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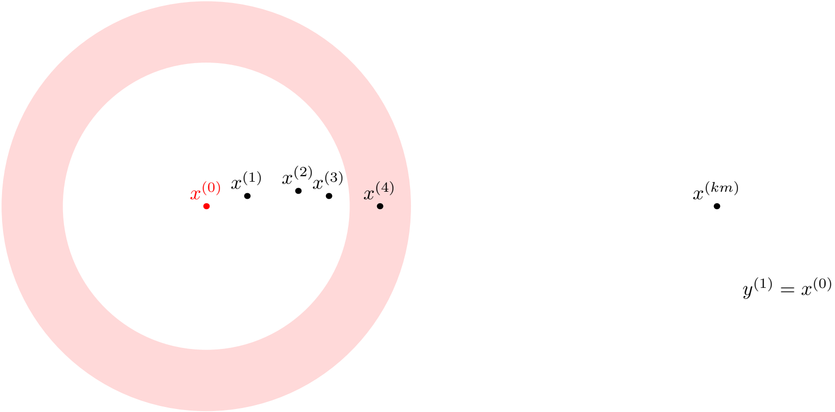

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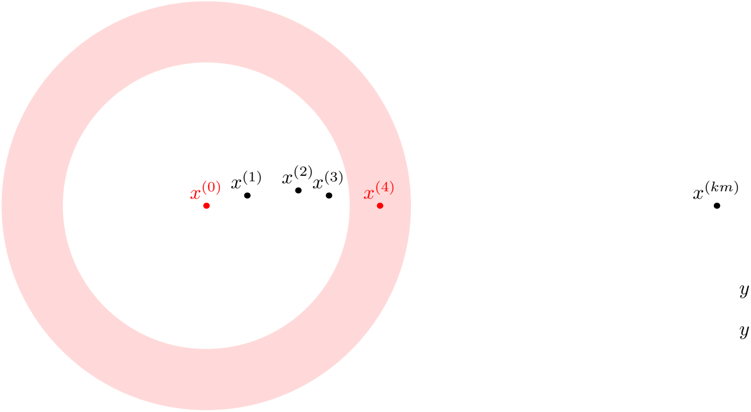

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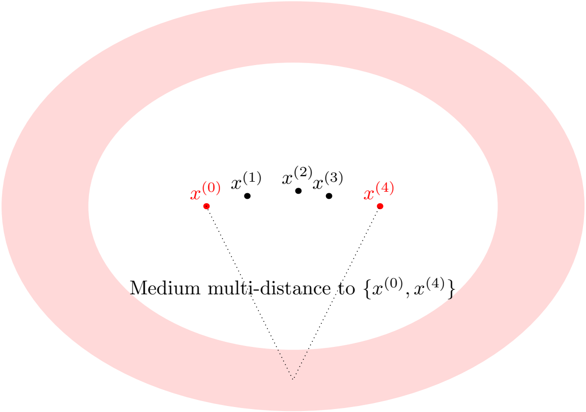

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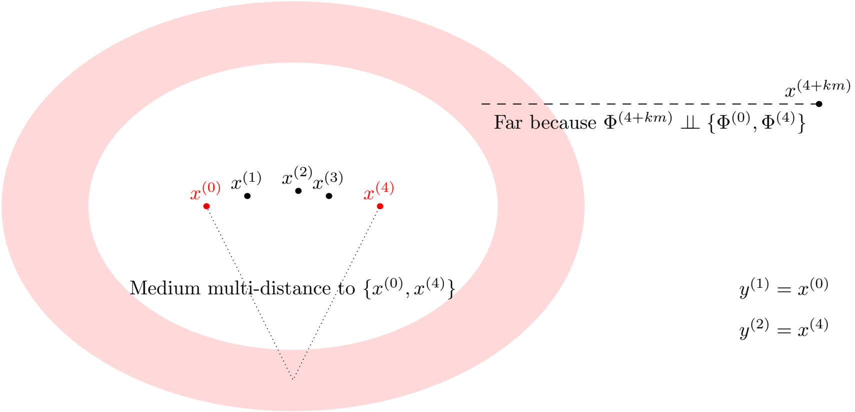

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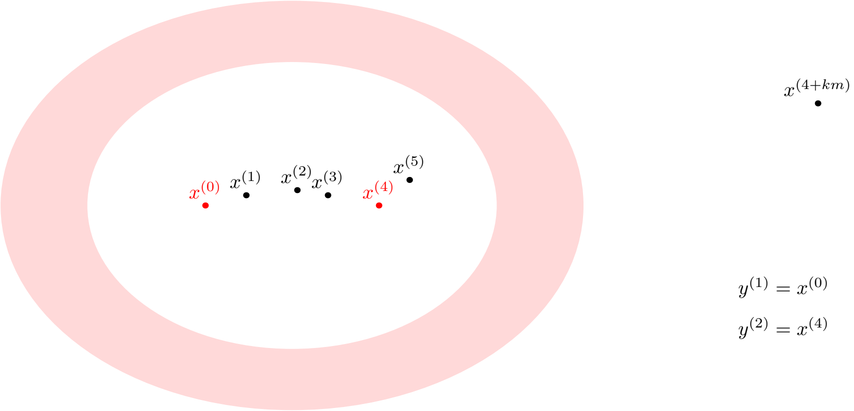

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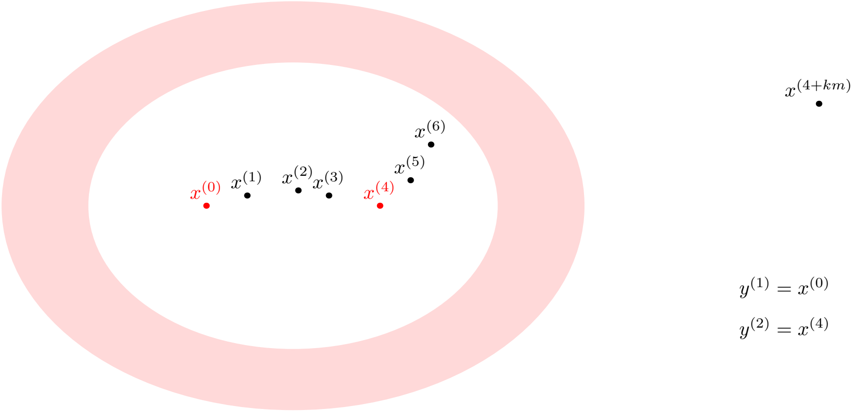

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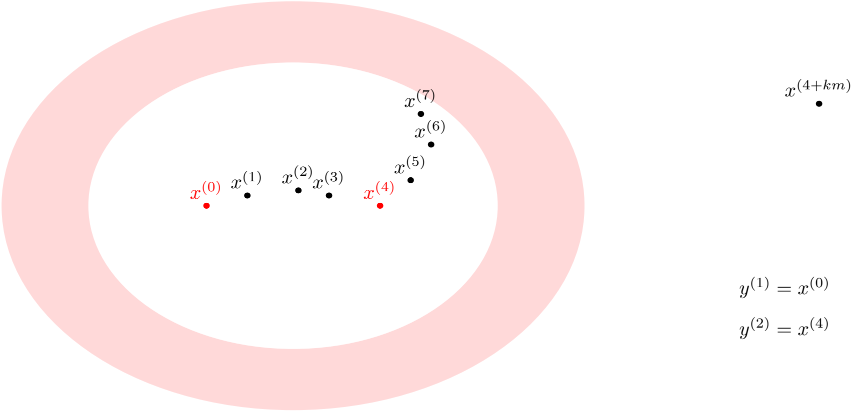

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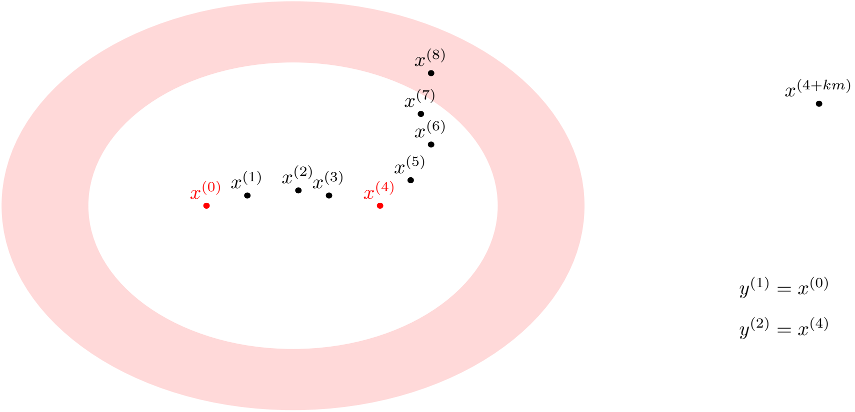

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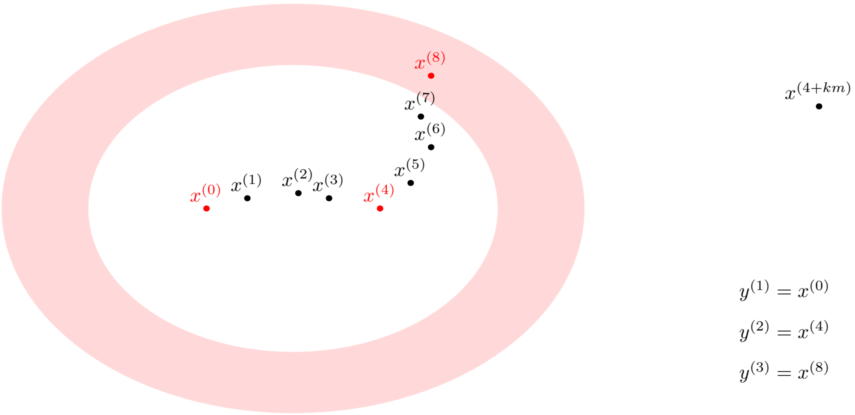

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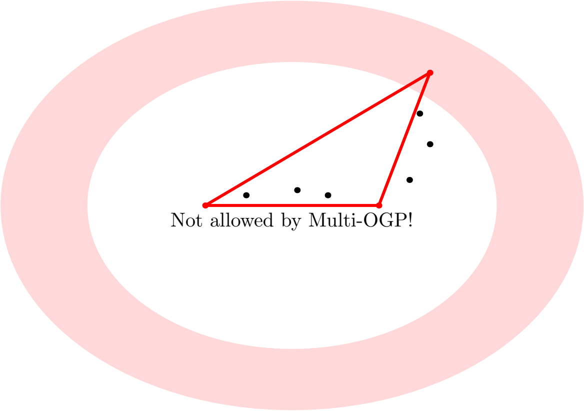

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Contradiction $\Rightarrow$ $\mathcal{A}$ cannot succeed.

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For us,

$$\mathbb{E}[\#\text{forbidden structure}] = (\text{entropic term})\mathbb{P}[y^{(1)}, \ldots, y^{(k)} \text{ all satisfying assignments}].$$

Main challenge: $\mathbb{P}$ term depends on $y^{(1)}, \ldots, y^{(k)}$ in complicated way.

How to choose condition so $\mathbb{P}$ term beats entropic term?

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Thank you!