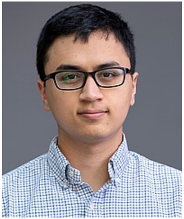
Algorithmic thresholds for random perceptron models

Brice Huang (Stanford → Yale)

Joint work with **Mark Sellke** (Harvard/OpenAI) and **Nike Sun** (MIT)



February 9, 2026 | Stanford probability seminar

Outline of talk:

Introduction

Main result: algorithmic threshold

Proof ideas for hardness, a simple case

Proof ideas for hardness, recap and generalization

Optimal algorithms via IAMP

Applications and conclusion

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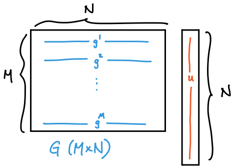

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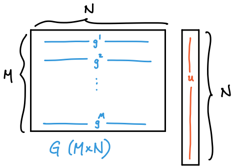

Formal statement (1D): let $\boldsymbol{G}$ be $M \times N$ iid gaussian with rows $\boldsymbol{g}^1, \ldots, \boldsymbol{g}^M$. For $\boldsymbol{u} \in \mathbb{S}^{N-1}$, **what are the possible measures**

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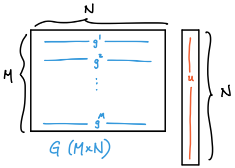

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$M/N \to \infty$: **all** low-dim proj $\to$ standard gaussian (DF84).
$M/N = \alpha \asymp 1$: $\exists$ non-gaussian projections (Bickel Kur Nadler 18).

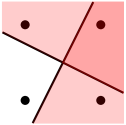
Statistical foundations of projection pursuit

A key challenge in analyzing high-dimensional data is to extract meaningful low-dimensional structures...with a limited amount of high-dimensional data, the results of projection pursuit and related ICA methods should be interpreted with great care...one may find projections with highly non-Gaussian components that have no statistical or scientific significance. (Bickel Kur Nadler 18)

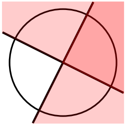
What is an "interesting" projection? We cannot expect universal agreement on what constitutes an "interesting" projection. A projection in which the data separate into disinct, meaningful clusters would certainly be interesting. But there are also interesting features that are not of the distinct cluster type (e.g. an edge, or jump of density, at the boundary of some region). (Huber 85)

The perceptron model

Perceptron: intersection of $\Sigma_N = \{-1, +1\}^N$ (**Ising**) or $\Sigma_N = N^{1/2}\mathbb{S}^{N-1}$ (**spherical**) with M iid random half-spaces:



Ising



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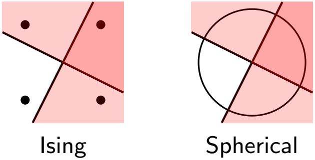

Formally, for $\boldsymbol{G} \in \mathbb{R}^{M \times N}$ iid gaussian, and $\kappa \in \mathbb{R}$ fixed,

$$S(\boldsymbol{G}) \equiv \left\{ \boldsymbol{x} \in \Sigma_N : \frac{(\boldsymbol{g}^a, \boldsymbol{x})}{\sqrt{N}} \geqslant \kappa \text{ for all } a \leqslant M \right\}.$$

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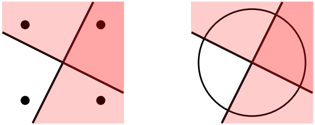

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Basic question: for what values of $\alpha = M/N$ is the random set $S(\boldsymbol{G})$ (non)empty whp?
$\leftrightarrow$ variant of projection pursuit problem, with $\boldsymbol{x} = \sqrt{N}\boldsymbol{u}$.

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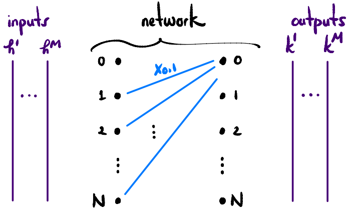
synaptic weights $\leftrightarrow$ $\boldsymbol{x}$, random patterns $\leftrightarrow$ rows $\boldsymbol{g}^a$ of $\boldsymbol{G}$.



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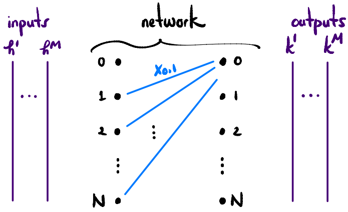

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• **Discrepancy** minimization: symmetric model with constraints $(\boldsymbol{g}^a, \boldsymbol{x})/N^{1/2} \in [-\kappa, \kappa]$ (Aubin Perkins Zdeborová 19; Spencer 85, Bansal 10, Lovett Meka 15, Rothvoss 17, Eldan Singh 18).

Prior work on statics of perceptron model

Recall: for $\boldsymbol{G} \in \mathbb{R}^{M \times N}$ iid gaussian, and $\kappa \in \mathbb{R}$ fixed,

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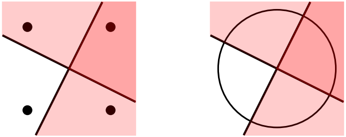
Many basic questions about typical size and structure of $S(\boldsymbol{G})$.



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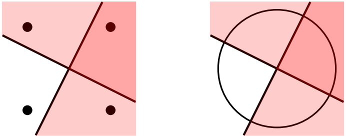

For example the **capacity problem**: does $S(\boldsymbol{G})$ go from nonempty to empty (whp) around a critical threshold $M/N \sim \alpha_{\text{cap}}$?

(Gardner Derrida 88, Krauth Mézard 89, Shcherbina Tirozzi 03, Kim Roche 98, Talagrand 00, Stojnic 13, Ding Sun 18, H. 24)

Prior work on statics of projection pursuit

Montanari Zhou 22: for $M/N = \alpha$, whp all feasible values of

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over $\boldsymbol{x} \in \sqrt{N}\mathbb{S}^{N-1}$ satisfy

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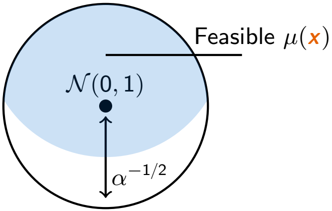

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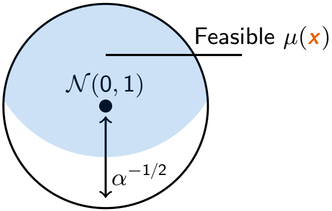

Montanari Zhou 24: conjectured description of feasible set via physics methods (in terms of Parisi-type formula)

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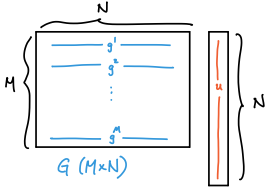
Can consider $\boldsymbol{x} \in \{-1, +1\}^N$ **(Ising)** of $\boldsymbol{x} \in \sqrt{N}\mathbb{S}^{N-1}$ **(spherical)**.

Algorithmic projection pursuit

Main result (HSS 25$^+$): for $M/N = \alpha$, we characterize the set of all measures EmpDist$(\boldsymbol{G}\boldsymbol{x}/\sqrt{N})$ achievable by a broad class of **algorithms** $\mathcal{A} : \boldsymbol{G} \mapsto \boldsymbol{x}$, for $\boldsymbol{x}$ in $\sqrt{N}\mathbb{S}^{N-1}$ or $\{-1,+1\}^N$.



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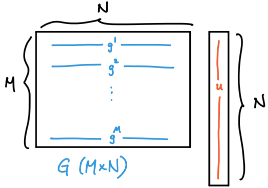

For $\boldsymbol{x} \in \sqrt{N}\mathbb{S}^{N-1}$, optimal algorithms obtained independently Montanari Zhou 24; our work gives matching hardness bound.

Theorem is phrased for **projection pursuit**, but implies algorithmic thresholds for **constraint satisfaction** and **optimization** settings.

Prior algorithmic results

Many works on both algorithms and hardness:

- Negative spherical perceptron ($\boldsymbol{I} = [\kappa, \infty)$ with $\kappa < 0$): Montanari Zhong Zhou 21, El Alaoui Sellke 20.

- Symmetric Ising perceptron ($\boldsymbol{I} = [-\kappa, \kappa]$): Bansal Spencer 20, Abbe Li Sly 21, Gamarnik Perkins Kızıldağ Xu 22+23.

- Asymmetric Ising perceptron ($\boldsymbol{I} = [\kappa, \infty)$): Kim Roche 98, Li Schramm Zhou 24.

We address these models in a unified way and close several gaps between algorithms and hardness results.

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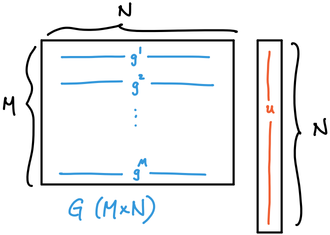

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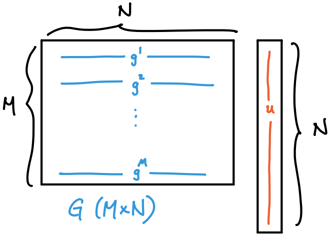

We want to characterize **the set of all possible measures**

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achievable by **algorithms** $\mathcal{A} : \boldsymbol{G} \mapsto \boldsymbol{x}$ (in $\sqrt{N}\mathbb{S}^{N-1}$ or $\{-1,+1\}^N$).

Informal result

Sharp algorithmic threshold for Lipschitz algorithms

$$\|\mathcal{A}(\boldsymbol{G}) - \mathcal{A}(\boldsymbol{G}')\|_2 \leqslant O(1)\|\boldsymbol{G} - \boldsymbol{G}'\|_2\,.$$

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Lipschitz algs include: grad desc, Langevin, AMP for $O(1)$ time, any $O(1)$-order method for $O(1)$ time, but not SOS or low-deg polynomials.

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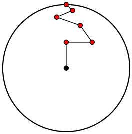

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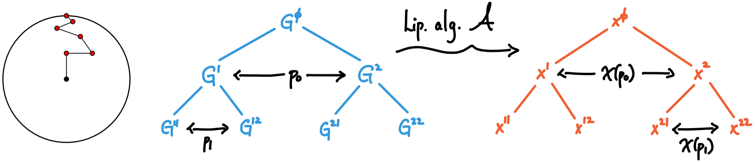

- Optimal **hardness** following **branching overlap gap property** framework (Gamarnik Sudan 14, H. Sellke 22).

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Main theorem (HSS 25^+)

For Lipschitz algorithms $\mathcal{A} : \boldsymbol{G} \mapsto \boldsymbol{x}$, the set $S^{\mathrm{alg}}(\alpha)$ of achievable

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is given by an **explicit stochastic control problem*** *(SDE subject to coefficient constraints), for $\boldsymbol{x}$ in $\sqrt{N}\mathbb{S}^{N-1}$ or $\{-1,+1\}^N$.*

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$$S^{\mathrm{alg}}(\alpha) = \left\{\mathrm{Law}(X_1) \; : \; X_1 = \int_0^1 b_t\, dt + \sigma_t\, dB_t, \right.$$
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*SDE will be shown in coming slides for various problem settings (spherical/Ising, symmetric/asymmetric).

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Consider the **1D controlled SDE** (with $X_0 = 0$, σ_t adapted)

$$\begin{cases} dX_t = \sigma_t \, dB_t \,, \text{ subject to} \\ \textbf{budget constraint } \mathbb{E}\Big[\big(\sigma_t - 1\big)^2\Big] \leqslant \dfrac{1}{\alpha} \text{ for all } 0 \leqslant t \leqslant 1 \,. \end{cases}$$

Simplified theorem for **symmetric** spherical perceptron

Simplest case: for spherical perceptron, $M/N = \alpha$, consider

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Theorem (HSS 25^+; **symmetric** spherical perceptron)

For the spherical perceptron, the measures $\mu^{\mathrm{sym}}(\boldsymbol{x})$ *achievable by Lipschitz algorithms are precisely of the form* $\mathrm{Law}(|X_1|)$ *for* X *satisfying the above controlled SDE.*

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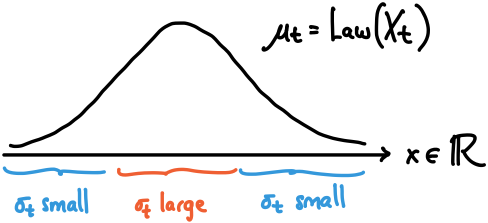

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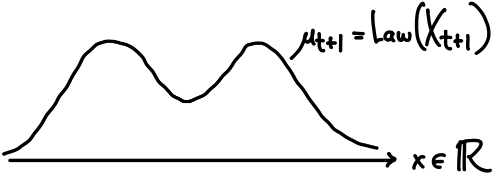

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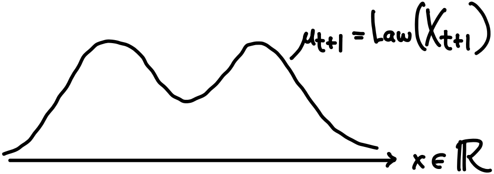

❗ We give a stochastic description, but the resulting evolution of $\mu_t = \mathrm{Law}(X_t)$ is **deterministic**.

General theorem for **(asymmetric)** spherical perceptron

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Consider 1D controlled SDE (with $X_0 = 0$, b_t and σ_t progressive)

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Analogous result for Ising perceptron (later).

Outline of talk:

Introduction

Main result: algorithmic threshold

Proof ideas for hardness, a simple case

Proof ideas for hardness, recap and generalization

Optimal algorithms via IAMP

Applications and conclusion

Method: analysis of Doob martingale

Recall: for Lipschitz algorithms $\mathcal{A} : \boldsymbol{G} \mapsto \boldsymbol{x}$, want to understand

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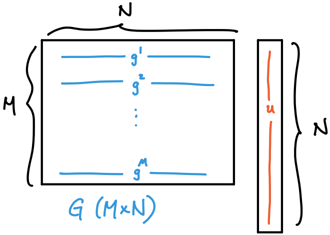

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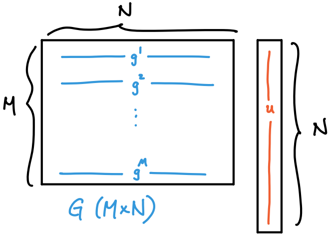

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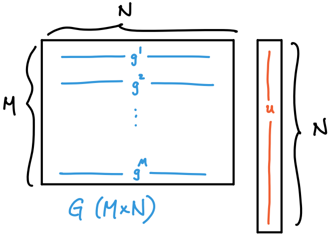

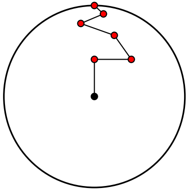

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Therefore, $\mu_1 = \mathrm{Law}(X_1) \leftrightarrow$ algorithmically achievable $\mu(\boldsymbol{x})$.

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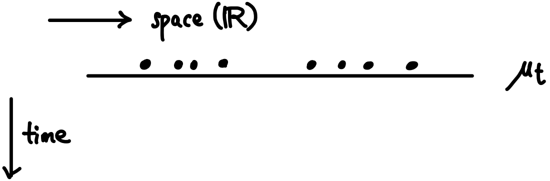

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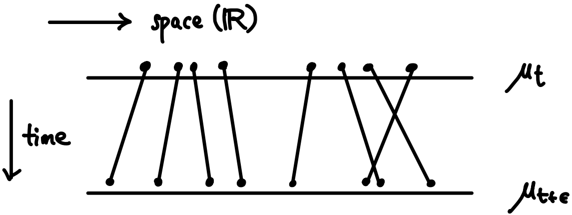

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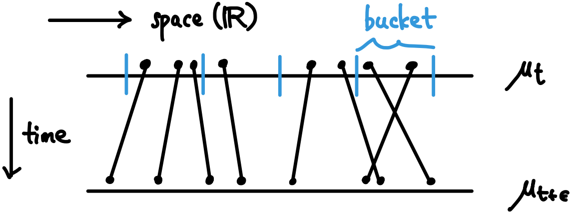

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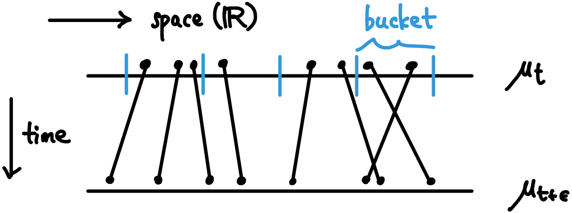

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! Matching Fokker–Planck eqns; **not** a pathwise description.

Symmetric spherical perceptron, single bucket

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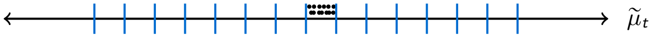

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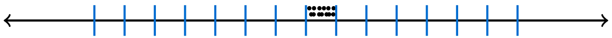

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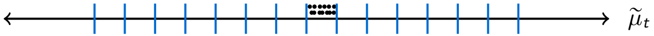

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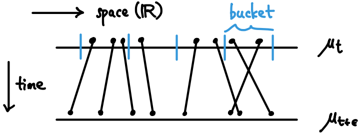
Symmetric spherical perceptron, many buckets



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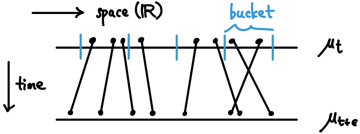

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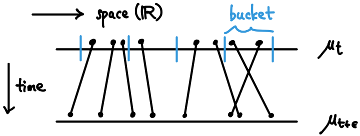

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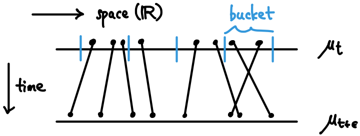
For any weights $\gamma_B \in \mathbb{R}$, **free probability** gives optimal value

$$\sum_B \gamma_B \mathrm{var}(B) \leqslant \left\| \sum_B \gamma_B \boldsymbol{W}^B \right\|_{\mathrm{op}} .$$

Symmetric spherical perceptron, many buckets



Tracking EmpDist($\boldsymbol{G}\boldsymbol{x}_t/\sqrt{N}$) with **multiple buckets**: $\boldsymbol{u} = \frac{\Delta \boldsymbol{x}_t}{\|\Delta \boldsymbol{x}_t\|}$,

$$\mathrm{var}(B) = \frac{1}{|B|}\sum_{a\in B}(\boldsymbol{g}^a, \boldsymbol{u})^2 = (\boldsymbol{u}, \boldsymbol{W}^B \boldsymbol{u})\,.$$

Want to characterize feasible tuples $(\mathrm{var}(B))_{\text{buckets } B}$

For any weights $\gamma_B \in \mathbb{R}$, **free probability** gives optimal value

$$\sum_B \gamma_B \mathrm{var}(B) \leqslant \left\| \sum_B \gamma_B \boldsymbol{W}^B \right\|_{\mathrm{op}}.$$

Intersection of all γ-constraints corresponds to **budget constraint**

$$\sum_B \frac{|B|}{M}\left(\sqrt{\mathrm{var}(B)} - 1\right)^2 \leqslant \frac{1}{\alpha}\,.$$

Symmetric spherical perceptron, SDE description

Recap: controlled SDE (with $X_0 = 0$, σ_t progressive)

$$\begin{cases} dX_t = \sigma_t \, dB_t \,, \text{ subject to} \\ \textbf{budget constraint } \mathbb{E}\Big[\big(\sigma_t - 1\big)^2\Big] \leqslant \dfrac{1}{\alpha} \text{ for all } 0 \leqslant t \leqslant 1 \,. \end{cases}$$

Theorem (HSS 25^+; **symmetric** spherical perceptron)

For the spherical perceptron, the measures $\mu^{\mathsf{sym}}(\boldsymbol{x})$ achievable by Lipschitz algorithms are precisely of the form $\mathrm{Law}(|X_1|)$ *for X satisfying the above controlled SDE.*

The last few slides explain the proof of the hardness bound.

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❗ Going from discretized processes to limiting stochastic control problems occupies large part of paper. See e.g. D. Lacker survey "Probabilistic compactification methods" for some of the ideas.

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General **(asymmetric)** spherical perceptron

For general spherical perceptron, track $\mu_t = \mathrm{EmpDist}(\boldsymbol{G}_t \boldsymbol{x}_t / \sqrt{N})$:

$$\begin{aligned}\Delta(\boldsymbol{G}_t \boldsymbol{x}_t) &= (\Delta \boldsymbol{G}_t)(\Delta \boldsymbol{x}_t) + \boldsymbol{G}_t(\Delta \boldsymbol{x}_t) + (\Delta \boldsymbol{G}_t)\boldsymbol{x}_t \\ &= \text{drift} + \text{diffusivity} + \text{fixed diffusivity}\,,\end{aligned}$$

with first two terms controlled by choice of $\Delta \boldsymbol{x}_t$.

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As before, **free probability** gives optimal value of

$$\sum_B \left\{ \boldsymbol{\theta}_B \mathrm{dr}(B)^2 + \boldsymbol{\varphi}_B \mathrm{cov}(B)^2 + \boldsymbol{\gamma}_B \mathrm{var}(B) \right\} \leqslant \left\| \textbf{spiked} \text{ random mtx} \right\|_{\mathrm{op}}.$$

Intersection of all $(\boldsymbol{\theta}, \boldsymbol{\varphi}, \boldsymbol{\gamma})$ constraints is our **budget constraint**.

General (asymmetric) spherical perceptron, recap

Recap: $p : [0,1] \to [0,1]$ increasing differentiable; controlled SDE

$$\begin{cases} dX_t = \sqrt{p'(t)}\; b_t\, dt + \sqrt{p(t) + tp'(t)}\; \sigma_t\, dB_t \\ \text{subject to the } \textbf{budget constraint} \\ \mathbb{E}\left[b_t^2 + \dfrac{p(t) + tp'(t)}{p(t)} (\sigma_t - 1)^2 \right] \leqslant \dfrac{1}{\alpha} \text{ for all } 0 \leqslant t \leqslant 1. \end{cases}$$

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In the previous slide we explained the basic proof strategy.
In the next slide we will clarify the role of the p function.

Radius schedule

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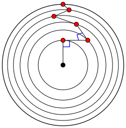
Interpretation of p: $p = \chi^{-1}$ where $\chi(t)$ is "radius schedule" of the Doob martingale $\boldsymbol{x}_t$:

$$\chi(t) = \frac{\mathbb{E}[\|\boldsymbol{x}_t\|^2]}{N}.$$



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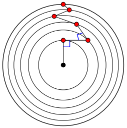

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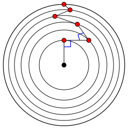

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If consider $\mu^{\mathrm{sym}}(\boldsymbol{x})$, suffices to take $b_t \equiv 0$ and $\chi \equiv 0$, i.e. $p \equiv 1$.

Ising perceptron

For Ising perceptron, we need to track joint evolution of

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Analogous result for **(asymmetric)** Ising perceptron, omitted here.

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Recall: Lipschitz algorithm $\mathcal{A} : \boldsymbol{G} \mapsto \boldsymbol{x}$.

- **Matrix Brownian motion** $\boldsymbol{G}_t$ with $\boldsymbol{G}_1 = \boldsymbol{G}$.
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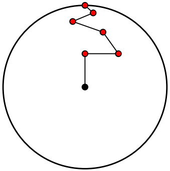

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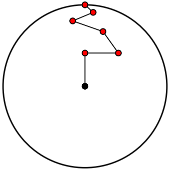
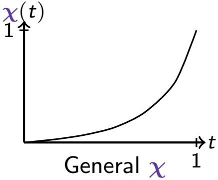
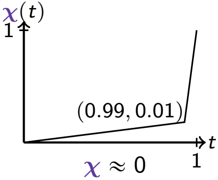
Process $\boldsymbol{x}_t$ has radius schedule $\chi(t) = \mathbb{E}(\|\boldsymbol{x}_t\|^2)/N = p^{-1}(t)$.







Matching IAMP algorithms

AMP: class of first-order algorithms

$$\begin{aligned} \boldsymbol{m}^{k+1} &= \boldsymbol{G}^{\top} f(\boldsymbol{n}^k, \ldots, \boldsymbol{n}^1) - \mathbf{ons}_{m,k} \,, \\ \boldsymbol{n}^k &= \boldsymbol{G} g(\boldsymbol{m}^k, \ldots, \boldsymbol{m}^1) - \mathbf{ons}_{n,k} \,. \end{aligned}$$

Limiting behavior well understood (Bolthausen 12, Bayati Montanari 11, many subsequent extensions).

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IAMP (Montanari 19, Montanari Zhou 24): for iterates $\boldsymbol{m}^k$, $\mathrm{EmpDist}(\boldsymbol{G}\boldsymbol{m}^k/\sqrt{N})$ simulates SDE with radius schedule $\chi \approx 0$, i.e. $p \approx 1$. Matches hardness result for symmetric perceptron.

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Our new IAMP simulates SDE for general $\chi = p^{-1}$: k-th step uses $\boldsymbol{G}_{t(k)}$, where $\boldsymbol{G}_t$ is matrix Brownian motion from hardness analysis. Matches hardness result for general (asymmetric) perceptron.

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Algorithmic bounds in constraint satisfaction settings

Constraint satisfaction: find $\boldsymbol{x}$ with $\boldsymbol{G}\boldsymbol{x}/\sqrt{N} \in \boldsymbol{I}$ coordinatewise.

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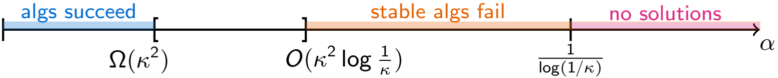

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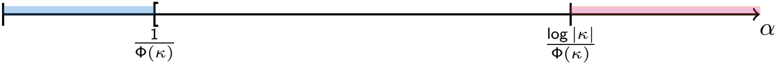

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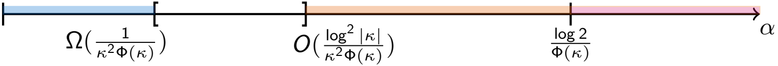

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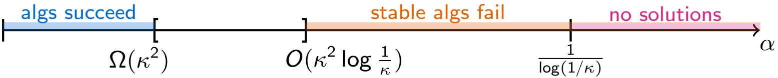

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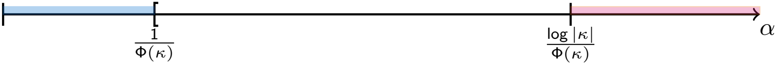

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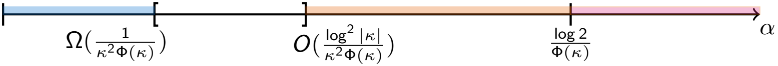

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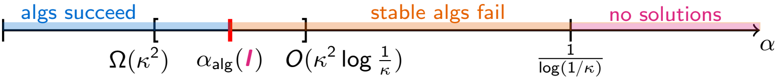

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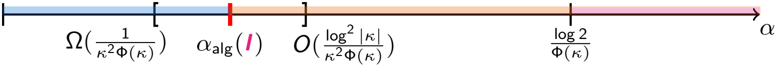

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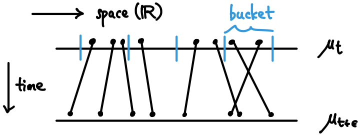

- Algorithms: new IAMP algorithm follows same SDE.

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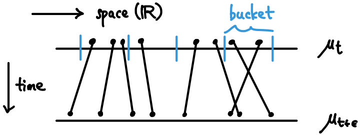

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Conclusion

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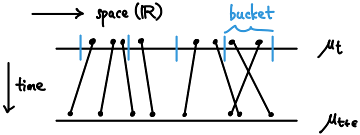

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Thank you!