Strong Topological Trivialization for Multi-Species Spin Glasses

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Harvard Probabilitas Seminar

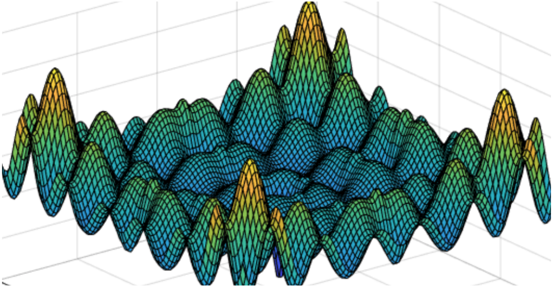
Joint work with Mark Sellke (Harvard)

Plan for this talk

1. Introduction and background
    - Landscape complexity of random functions
    - Topological trivialization
2. Strong topological trivialization for spin glasses
3. Multi-species spin glasses

Random Landscapes

Landscapes of random, high-dimensional functions



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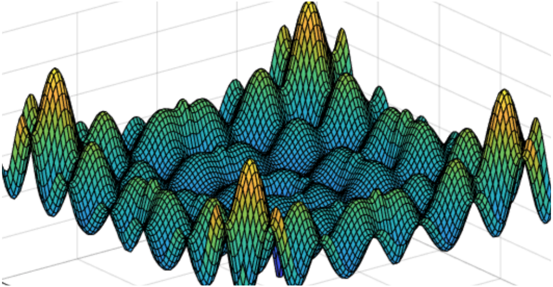

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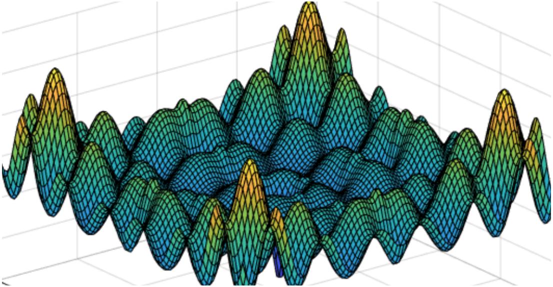

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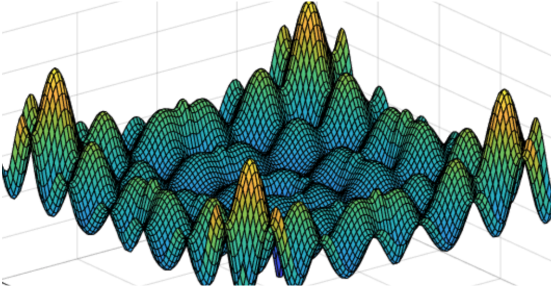

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How complicated is the landscape of a random function?

Mean Field Spin Glasses

Polynomials $H_N : \mathbb{R}^N \to \mathbb{R}$ with **random** coefficients, e.g. random cubic

$$H_N(\boldsymbol{\sigma}) = \frac{1}{N} \sum_{i_1, i_2, i_3 = 1}^{N} g_{i_1, i_2, i_3} \cdot \sigma_{i_1} \sigma_{i_2} \sigma_{i_3} \qquad g_{i_1, i_2, i_3} \underset{\text{i.i.d.}}{\sim} \mathcal{N}(0, 1)$$

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More generally, mix different degrees. For $\gamma_1, \gamma_2, \ldots \geq 0$,

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Computes expected number of zeros of random function $f : \Omega \times [0, L] \to \mathbb{R}$

$$\mathbb{E}[\#\text{zeros}(f)] = \int_{[0,L]} \mathbb{E}\left[|f'(x)| \middle| f(x) = 0\right] \varphi_{f(x)}(0)\, \mathrm{d}x.$$

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For multi-dimensional $f : \Omega \times [0, L]^N \to \mathbb{R}^N$:

$$\mathbb{E}[\#\text{zeros}(f)] = \int_{[0,L]^N} \mathbb{E}\left[|\det \nabla f(\boldsymbol{x})|\middle|f(\boldsymbol{x}) = 0\right] \varphi_{f(\boldsymbol{x})}(0) \,\mathrm{d}\boldsymbol{x}.$$

A simple example

How many zeros does this function have on $[-L, L]$, $L < 1$, in expectation?

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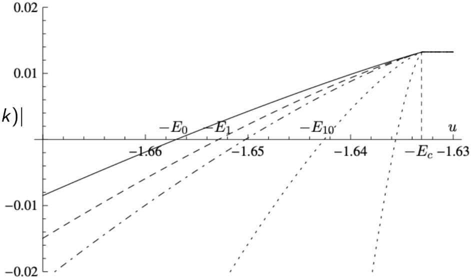
- Other applications:
  - (Sagun-Guney-Ben Arous-LeCun 14) neural networks
  - (Ben Arous-Mei-Montanari-Nica 19) spiked tensor models
  - (Ben Arous-Fyodorov-Khoruzhenko 21, Subag 23) non-gradient vector fields
  - (Ben Arous-Bourgarde-McKenna 23) elastic manifold

Critical Point Complexity in Pure Spin Glasses

- (Auffinger-Ben Arous-Černý 13): $\mathbb{E}\#$ critical points at given (energy, index)

$\frac{1}{N} \log \mathbb{E}|\mathrm{Crt}(E,k)|$

E_k index k



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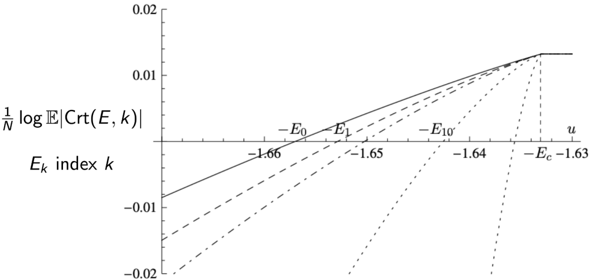

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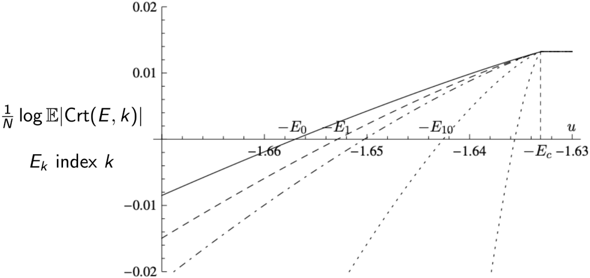

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- Consequence: ground state energy matching Parisi formula

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- $\xi'(1) > \xi''(1)$ equivalent to $\gamma_1^2 > \sum_{p\geq 3} p(p-2)\gamma_p^2$
- $\mathbb{E}|\mathrm{Crt}| = 2 + o(1)$ achieved using exact formula for GOE
    - If Kac-Rice random matrix not GOE, current tools only achieve $\mathbb{E}|\mathrm{Crt}| = e^{o(N)}$

Guiding Questions

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- Does trivialization have algorithmic implications, e.g. fast convergence of Langevin dynamics?
  - Or can regions of small, nonzero gradient still obstruct algorithms?

Guiding Questions

- Do phase boundaries of annealed and quenched trivialization coincide?
  - Or can $|\mathrm{Crt}| = 2$ w.h.p. in the regime where $\mathbb{E}|\mathrm{Crt}| \geq e^{cN}$?
- Does trivialization have algorithmic implications, e.g. fast convergence of Langevin dynamics?
  - Or can regions of small, nonzero gradient still obstruct algorithms?
- Can we boost $\mathbb{E}|\mathrm{Crt}| = e^{o(N)}$ to $|\mathrm{Crt}| = 2$ w.h.p. without exact formulas?

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Theorem (H.-Sellke 23)

If H is STT, then Langevin dynamics at low enough temperature mixes in $O(\log N)$ time in TV.

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- In these models, phase boundary for *annealed* trivialization also new

Annealed Critical Point Complexity

Kac-Rice on $\nabla_{\mathsf{sp}} H$:

$$\mathbb{E}|\mathrm{Crt}| = \int_{S_N} \mathbb{E}\left[|\det \nabla_{\mathsf{sp}}^2 H(\boldsymbol{\sigma})| \middle| \nabla_{\mathsf{sp}} H(\boldsymbol{\sigma}) = 0 \right] \varphi_{\nabla_{\mathsf{sp}} H(\boldsymbol{\sigma})}(0) \, \mathrm{d}\boldsymbol{\sigma}.$$

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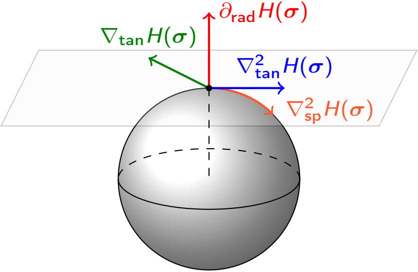
Here ∇_{sp}, ∇^2_{sp} are **Riemannian gradient, Hessian**

$$\nabla_{\mathrm{sp}} H(\boldsymbol{\sigma}) = \nabla_{\tan} H(\boldsymbol{\sigma}), \qquad \nabla^2_{\mathrm{sp}} H(\boldsymbol{\sigma}) = \nabla^2_{\tan} H(\boldsymbol{\sigma}) - \underbrace{\partial_{\mathrm{rad}} H(\boldsymbol{\sigma}) I_{N-1}}_{\text{curvature term}}.$$



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$$\partial_{\mathrm{rad}} H(\boldsymbol{\sigma}) =_d c_1 \mathcal{N}(0, 1/N) \qquad \nabla^2_{\tan} H(\boldsymbol{\sigma}) =_d c_2 \mathrm{GOE}_{N-1}$$

$$\nabla_{\tan} H(\boldsymbol{\sigma}) =_d c_3 \mathcal{N}(0, I_{N-1})$$

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Integrate out $\partial_{\mathrm{rad}} H(\boldsymbol{\sigma}) = x$:

$$\mathbb{E}|\mathrm{Crt}| = (\text{simple term}) \times \int_{\mathbb{R}} \mathbb{E}\left|\det(c_2 \mathrm{GOE}_{N-1} - xI)\right| \exp\left(-\frac{Nx^2}{2c_1}\right) \,\mathrm{d}x$$

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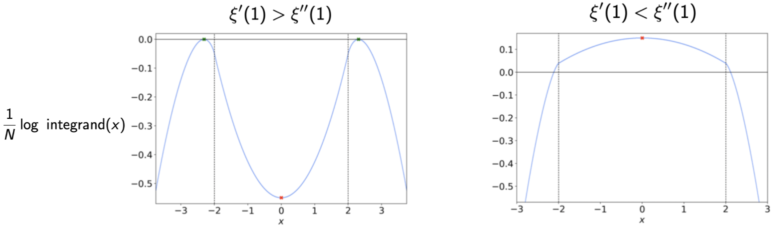

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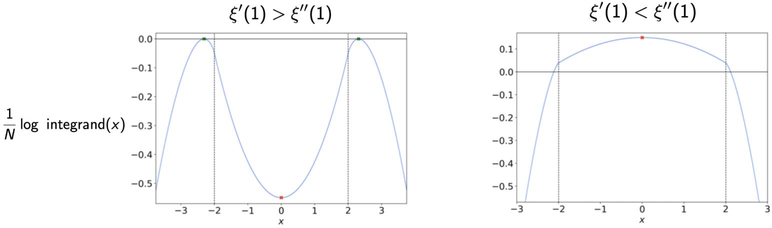

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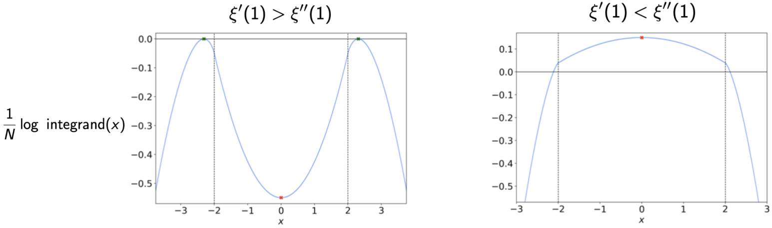
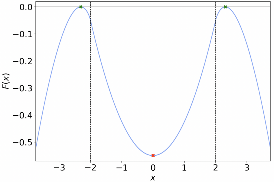


- $\xi'(1) > \xi''(1) \Rightarrow \mathbb{E}|\mathrm{Crt}| = e^{o(N)}$, (weak form of) annealed trivialization
- $\xi'(1) < \xi''(1) \Rightarrow \mathbb{E}|\mathrm{Crt}| \geq e^{cN}$, failure of annealed trivialization

Statistics of Critical Points

Suppose we are in trivial regime $\xi'(1) > \xi''(1)$, so $\mathbb{E}|\mathrm{Crt}| = e^{o(N)}$



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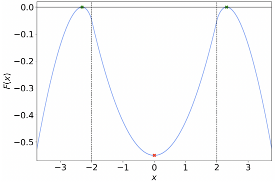

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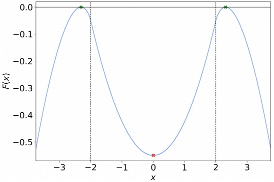

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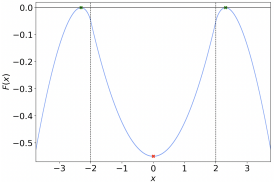

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Attempt: from $e^{o(N)}$ Critical Points to 2

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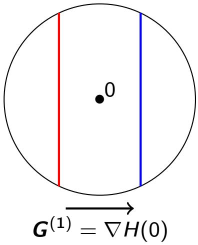

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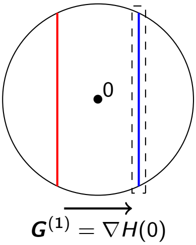

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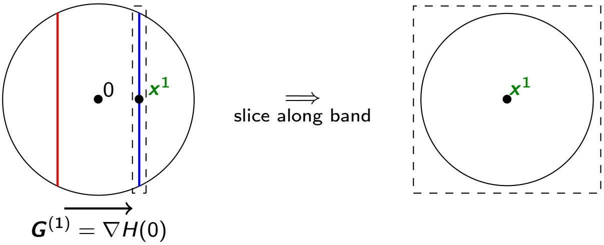

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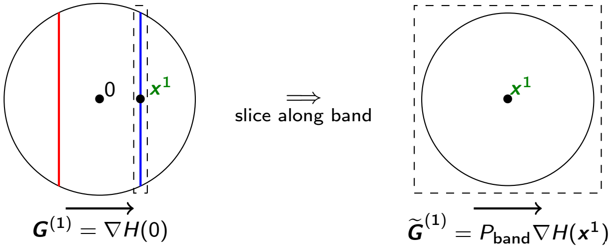

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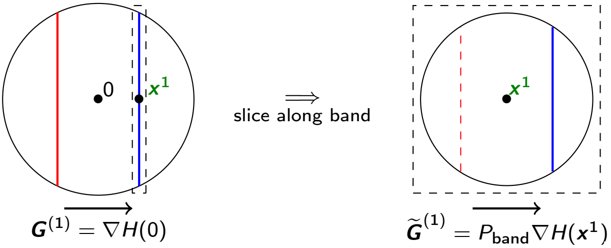

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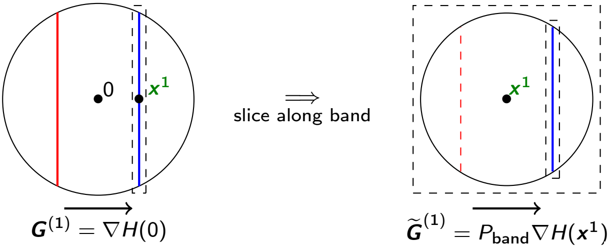

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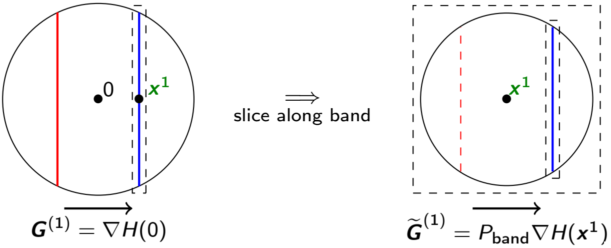

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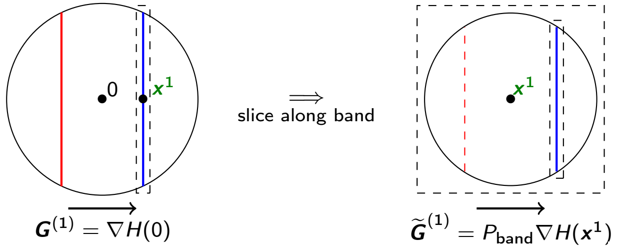

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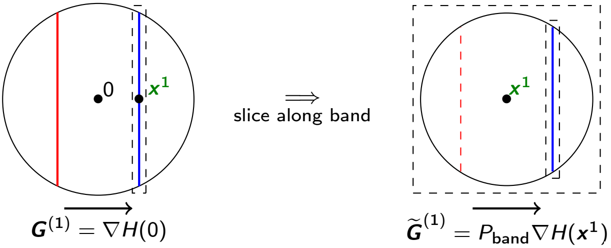

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- **This does not work!** Critical points brittle, cannot tolerate $o(1)$ error ☹

Approximate to Exact Critical Points

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With probability $1 - e^{-cN}$, *all* ε*-approximate critical points* $\boldsymbol{\sigma}$ *of* H *satisfy:*

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and $\nabla^2_{\text{sp}} H(\boldsymbol{\sigma})$ *is well-conditioned.*

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Band Recursion and Strong Trivialization

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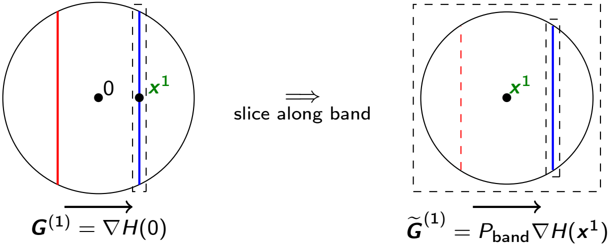

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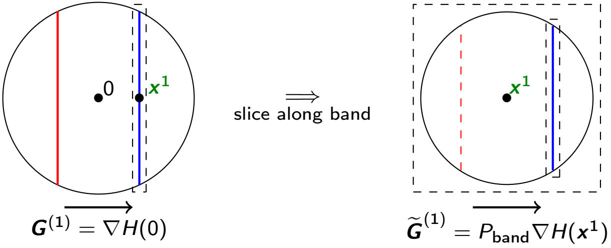

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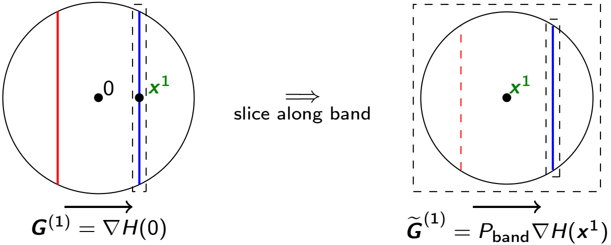

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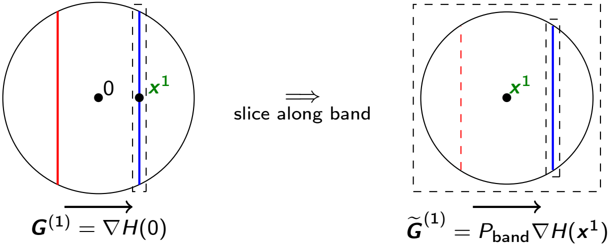

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- So 2 exact crits, all approx crits near an exact crit. **Strong trivialization**!

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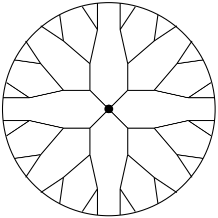

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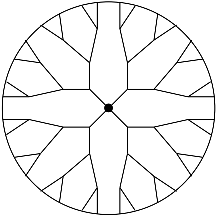

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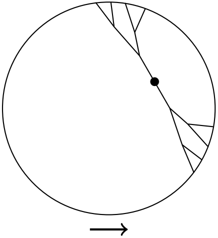

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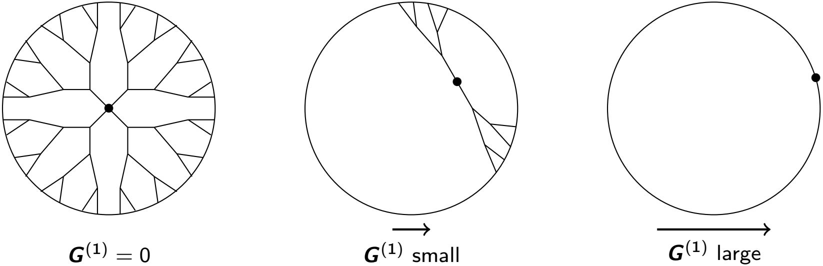

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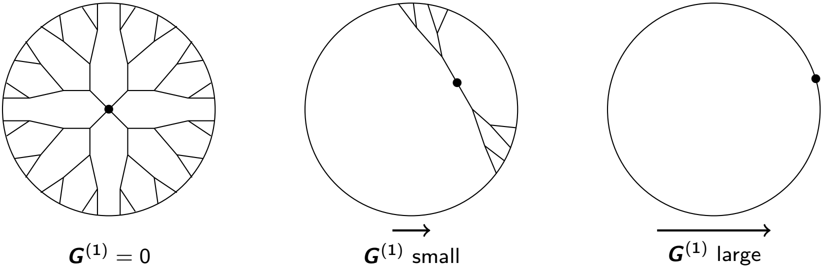

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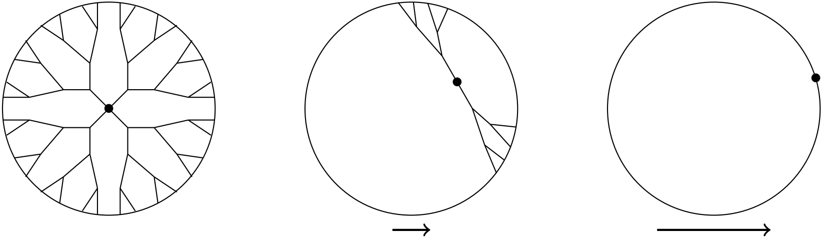

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- AMP constructs e^{cN} points in algorithmic tree, all approximate critical points

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- Example: **bipartite SK** $H(\boldsymbol{x}, \boldsymbol{y}) = \langle \boldsymbol{x}, \boldsymbol{G}\boldsymbol{y} \rangle$ where $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{S}_{N/2}$

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$\xi(q_1, \ldots, q_r)$ now r-variate polynomial (so $\nabla\xi \in \mathbb{R}^r$, $\nabla^2\xi \in \mathbb{R}^{r\times r}$)

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If $\mathrm{diag}(\nabla\xi(\vec{1})) \not\succ \nabla^2\xi(\vec{1})$, *both annealed and strong trivialization fail.*

- $\mathbb{E}|\mathrm{Crt}| \geq e^{cN}$
- e^{cN} *well-separated approximate critical points w.h.p.*

Annealed Complexity for Multi-Species

$$\begin{aligned}\mathbb{E}|\mathrm{Crt}| &= (\text{simple term}) \times \mathbb{E}|\det \nabla^2_{\mathrm{sp}} H(\boldsymbol{\sigma})| \\ &= (\text{simple term}) \times \mathbb{E}|\det \left(\nabla^2_{\tan} H(\boldsymbol{\sigma}) - \mathrm{Curv}\right)|\end{aligned}$$

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Laplace's principle $\Rightarrow$ maximize exponential rate of integrand over $\vec{x} \in \mathbb{R}^r$

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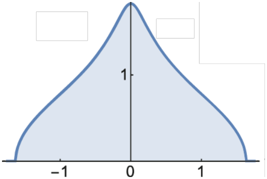

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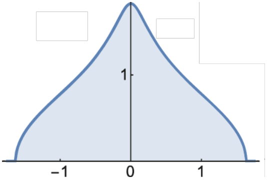

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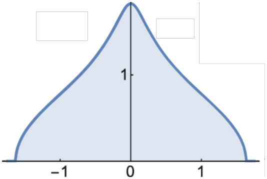

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- (Ben Arous-Bourgade-McKenna 23, McKenna 21): this is correct **to leading exponential order**

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Annealed Trivialization: Solving the Variational Problem

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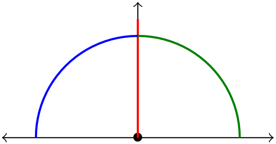

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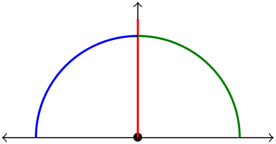

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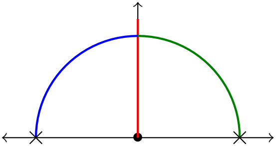

- About 3^r solutions, one for each choice of $\Re(m_s(0)) = 0, < 0, > 0$
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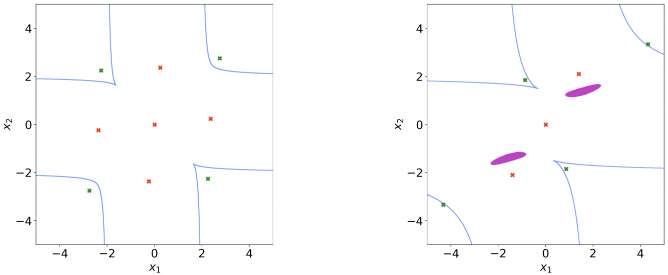

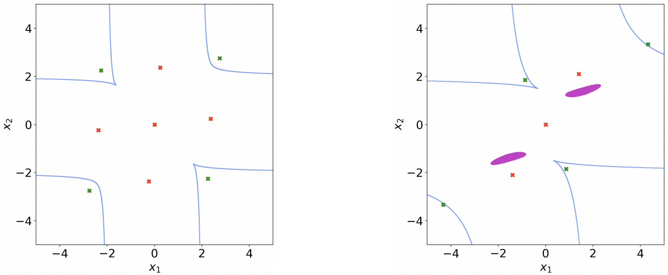

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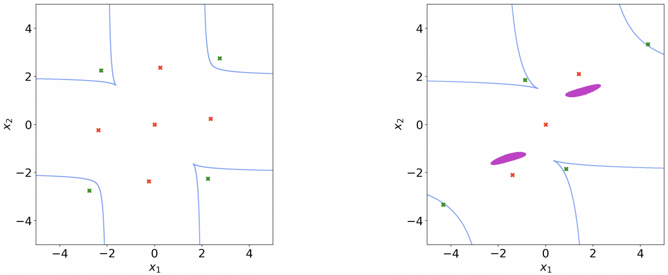

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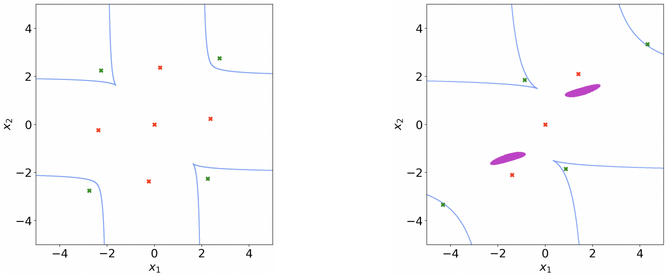

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- Thus $\max F = 0 \Rightarrow \mathbb{E}|\mathrm{Crt}| = e^{o(N)}$

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- Converse: algorithmically construct e^{cN} approx crits with AMP

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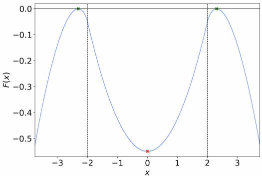

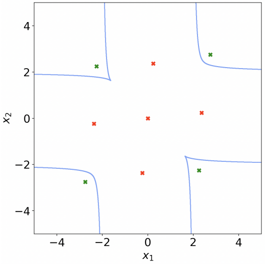

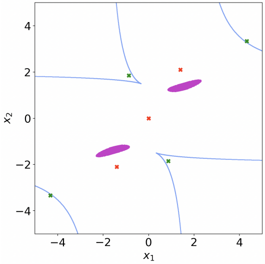

Thank you!