On zeros and algorithms for disordered systems: mean field spin glasses

Brice Huang (Stanford → Yale)

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NSF

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Input: matrix $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ (**interactions**) and $\beta \geqslant 0$ (**inv temperature**)

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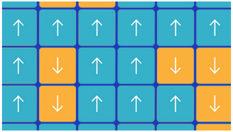
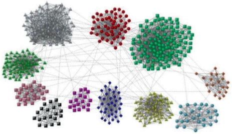
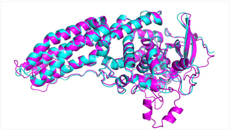
Applications to physics, statistics, ML, comp bio, ...

Counting and sampling

Counting: estimate **partition fn**

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When are these problems solvable in poly time, and when are they hard?

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Key insight: computational complexity related to **phase transitions**

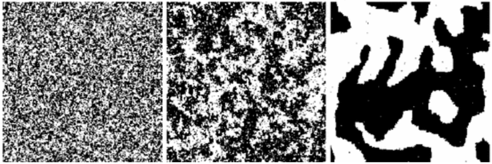

Image from Mora Urbina Cortez Rica 15

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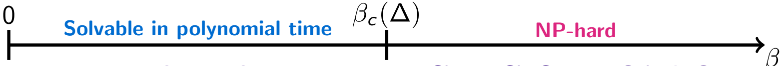

Weitz 06, Sinclair Srivastava Thurley 14, Chen Liu Vigoda 20+21

Sly 10, Sly Sun 12, Galanis Ge Stefankovic Vigoda Yin 14 Galanis Stefankovic Vigoda 15+16

Beyond worst-case

Main question of this work

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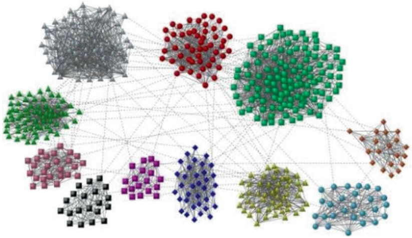
TCS motivation: average case complexity, beyond worst-case analysis

Statistics motivation: Bayesian posterior distributions are random



Physics motivation: models of disordered systems ("spin glasses")

Sherrington–Kirkpatrick model

SK model: $\boldsymbol{A} \sim \mathrm{GOE}(n)$ — that is, $A_{i,i} \sim \mathcal{N}(0, \frac{2}{n})$, $A_{i,j} = A_{j,i} \sim \mathcal{N}(0, \frac{1}{n})$

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Main result

Main theorem for SK model (Bencs H Lee Liu Regts 25)

For any $\varepsilon > 0$, there is an $n^{O(\log(n/\varepsilon))}$ time algorithm which, given $\boldsymbol{A}$ and $\beta \in [0, 1)$, outputs $\widehat{Z}$ such that

$$\mathbb{P}_{\boldsymbol{A} \sim \mathrm{GOE}(n)} \left[\frac{\widehat{Z}}{Z_{\boldsymbol{A}}(\beta)} \in [1 - \varepsilon, 1 + \varepsilon] \right] = 1 - o_n(1)\,.$$

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Aizenman Lebowitz Ruelle 87 implies case where ε **independent** of n.
Take $\widehat{Z}$ low-degree cluster expansion of $\boldsymbol{A}$.

Our result: arbitrary accuracy ε, can depend on n

More generally: (mixed) p-spin models

e.g. 3-spin model: $H(\boldsymbol{x}) = \sum_{i,j,k=1}^{n} g_{i,j,k} x_i x_j x_k$, Gibbs distr $\mu(\boldsymbol{x}) = \frac{1}{Z} e^{\beta H(\boldsymbol{x})}$

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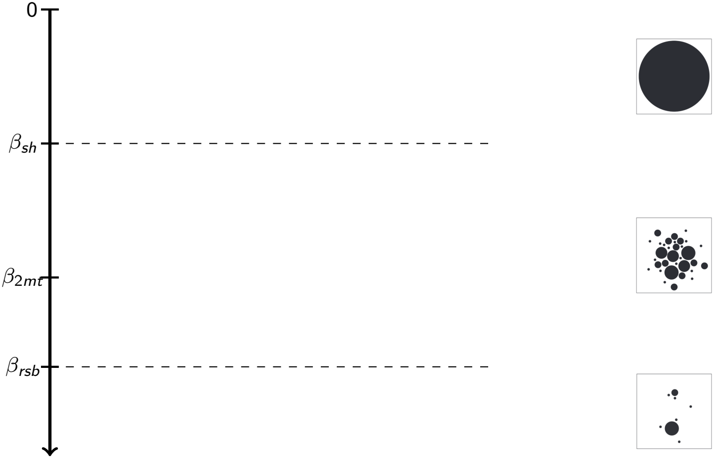

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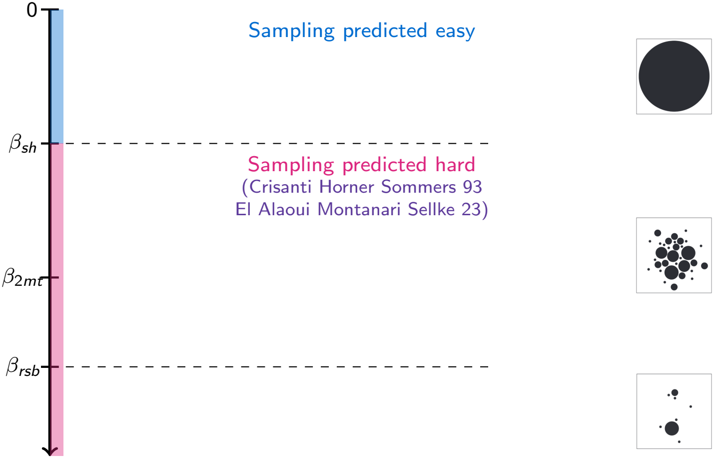

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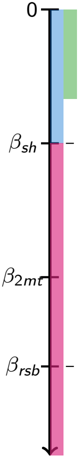

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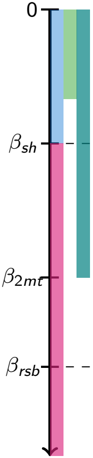

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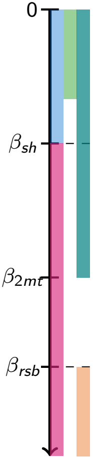

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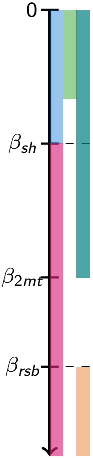

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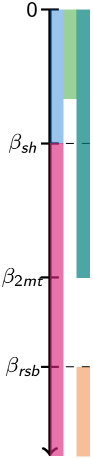

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Second moment threshold β_{2mt}: max β where $\mathbb{E}[Z_{\boldsymbol{A}}(\beta)^2] = O(1) \cdot \mathbb{E}[Z_{\boldsymbol{A}}(\beta)]^2$

Proof method

Barvinok's interpolation method: if $Z_G(\beta)$ has no zeros in **complex** disk $\mathbb{D}(0, r)$, can estimate $\log Z_G(\beta)$ for $|\beta| < r$ to arbitrary accuracy by truncating its Taylor series

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Theorem: zero-freeness of SK model (Bencs H Lee Liu Regts 25)

For any (constant) $r < 1$,

$$\mathbb{P}_{\boldsymbol{A}\sim\mathrm{GOE}(n)}\left[Z_{\boldsymbol{A}}(\beta) \text{ has no zeros in } \mathbb{D}(0,r)\right] = 1 - o_n(1)\,.$$

Jensen's formula for counting zeros

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For $R > 0$ and analytic $f : \mathbb{C} \to \mathbb{C}$ with $f(0) \neq 0$,

$$\sum_{\substack{\omega \in \mathbb{D}(0,R) \\ f(\omega)=0}} \log\left(\frac{R}{|\omega|}\right) = \mathop{\mathbb{E}}_{\theta \sim \mathsf{unif}([0,2\pi])}\left[\log \frac{|f(Re^{i\theta})|}{|f(0)|}\right]$$

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Set $R = 1 - \frac{\eta}{2}$, $r = 1 - \eta$; $M = \#\{\text{zeros of } Z_{\boldsymbol{A}}(\beta) \text{ in } \mathbb{D}(0, r)\}$

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$$\left(\log \frac{R}{r}\right) M \leqslant \qquad \mathbb{E}_\theta\left[\log \frac{|Z_{\boldsymbol{A}}(Re^{i\theta})|}{|Z_{\boldsymbol{A}}(0)|}\right]$$

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For $R > 0$ and analytic $f : \mathbb{C} \to \mathbb{C}$ with $f(0) \neq 0$,

$$\sum_{\substack{\omega \in \mathbb{D}(0,R) \\ f(\omega)=0}} \log\left(\frac{R}{|\omega|}\right) = \mathop{\mathbb{E}}_{\theta \sim \mathrm{unif}([0,2\pi])}\left[\log \frac{|f(Re^{i\theta})|}{|f(0)|}\right]$$

Set $R = 1 - \frac{\eta}{2}$, $r = 1 - \eta$; $M = \#\{\text{zeros of } Z_{\boldsymbol{A}}(\beta) \text{ in } \mathbb{D}(0,r)\}$

$$\mathbb{E}_{\boldsymbol{A}}\left(\log \frac{R}{r}\right) M \leqslant \quad \mathbb{E}_{\boldsymbol{A}}\mathbb{E}_{\theta}\left[\log \frac{|Z_{\boldsymbol{A}}(Re^{i\theta})|}{|Z_{\boldsymbol{A}}(0)|}\right]$$

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From $O(1)$ to $o(1)$ zeros

Consequence of Jensen's formula: for all constant $\eta > 0$,

$$\mathbb{E}_{\boldsymbol{A}}\#\Big\{\text{zeros of } Z_{\boldsymbol{A}}(\beta) \text{ in } \mathbb{D}(0, 1-\eta)\Big\} = O_\eta(1)$$

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Re-run the same Jensen's formula strategy with

$$X_{\boldsymbol{A}}(\beta) = Z_{\boldsymbol{A}}(\beta) \cdot Y_{\boldsymbol{A}}(\beta)$$

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We give **quasi-polynomial algorithm** to approximate a mean-field spin glass partition function to **arbitrary accuracy**, whp

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Mixed *p*-spin model: works to 2nd moment threshold β_{2mt}.
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Future directions:

- **polynomial-time** algorithm for counting?
- **sparse models**? e.g. hardcore model on random *d*-reg graph
- stronger evidence of **hardness** beyond β_c?

Thank you!