Strong low degree hardness in random optimization problems

Brice Huang (Stanford → Yale)

Joint work with Mark Sellke (Harvard / OpenAI)

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Outline of talk:

Introduction and motivating problems

Overlap gap property: the basics

Hardness for low degree polynomials

Strong low degree hardness

Other OGPs and SLDH

Further applications and conclusion

Random optimization problems

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- **Random perceptron**: given IID $\boldsymbol{g}^1, \dots, \boldsymbol{g}^M \sim \mathcal{N}(0, I_N)$, $\underset{\substack{\uparrow \\ \text{activation}}}{\varphi} : \mathbb{R} \to \mathbb{R}$,

$$H(\boldsymbol{\sigma}) = \sum_{a=1}^{M} \varphi\big(\langle \boldsymbol{\sigma}, \boldsymbol{g}^a \rangle\big)$$

Maximize over spherical domain $\boldsymbol{\sigma} \in \mathbb{S}^{N-1}$

Random optimization problems: motivation

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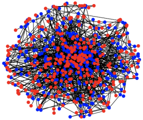

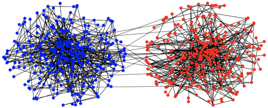

(Mossel Neeman Sly 12, Abbe Bandeira Hall 15, Mossel Sly Sohn 23)

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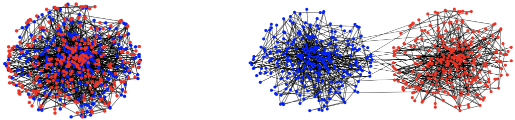

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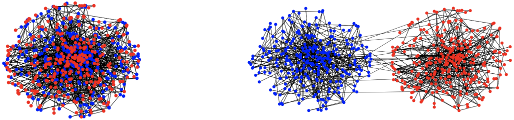

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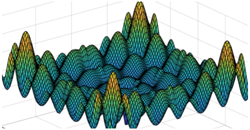

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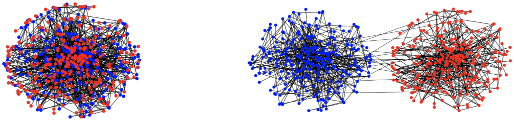

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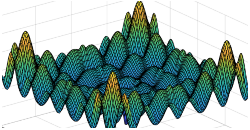

Random perceptron ↔ loss landscape of neural net on **random data**.

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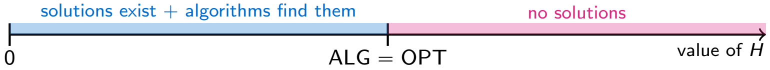

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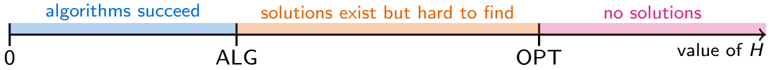

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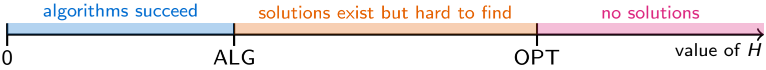

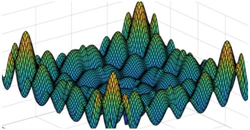

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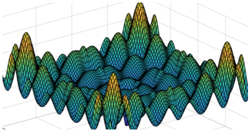

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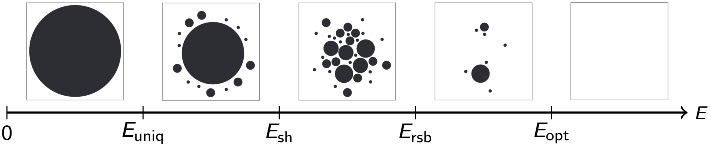
- **Average case** setting — how to reason about algorithmic hardness?

Solution geometry and algorithmic connections

Level sets $\{\boldsymbol{\sigma} : H(\boldsymbol{\sigma}) \approx E\}$ exhibit (conjectured) phase transitions:



(Image from Krzakala Montanari Ricci-Tersenghi Semerjian Zdeborová 06)

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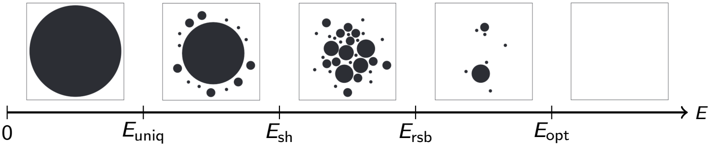

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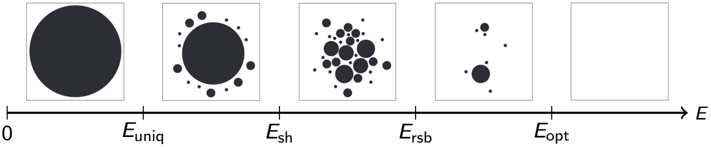

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Does solution geometry have rigorous implications for algorithmic tractability?

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Stability: view algorithm $\mathcal{A} : H \to \boldsymbol{\sigma}$ as function of problem instance H

Basic idea is to argue:

- If stable algorithm $\mathcal{A}$ succeeds, it can build a solution constellation with some prescribed geometry
- But we can show this constellation doesn't exist in solution space

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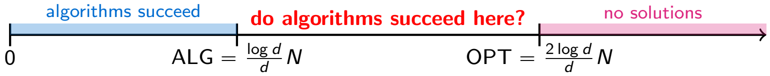

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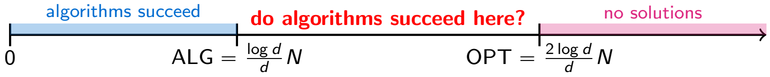

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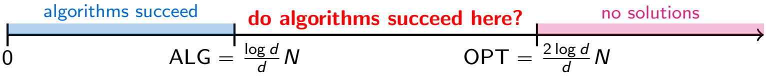

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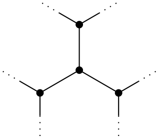

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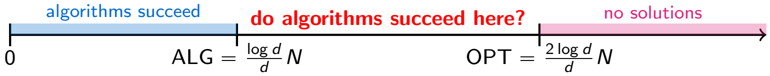

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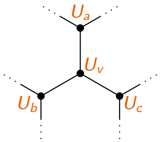

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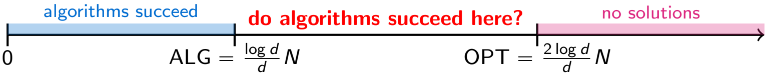

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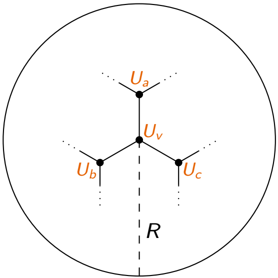

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At each $v \in G$, decide output $\sigma_v \in \{0,1\}$ based on only data within R-**neighborhood** of v ($R = O(1)$)

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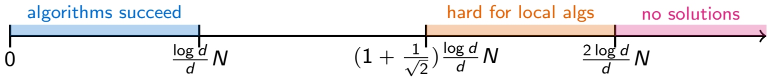
Theorem (Gamarnik Sudan 14)

Any $O(1)$-local algorithm does **not** *find an independent set of size $\geqslant (\mathbf{1 + \frac{1}{\sqrt{2}}} + o_d(1))\frac{\log d}{d}N$ in $G(N, d/N)$, whp.*



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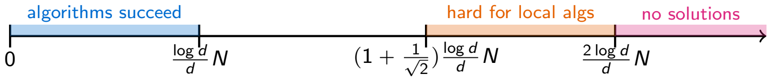

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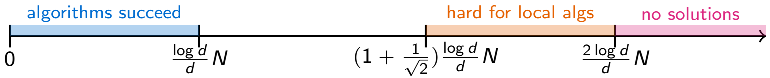

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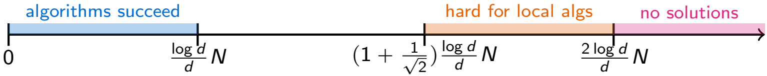

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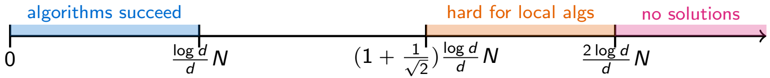

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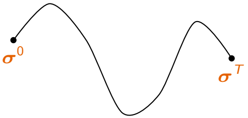

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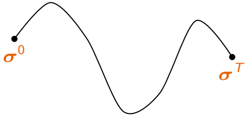

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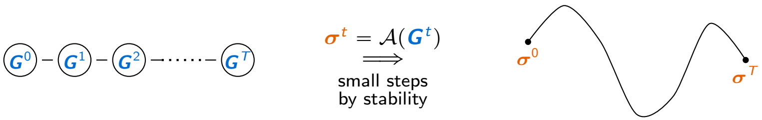

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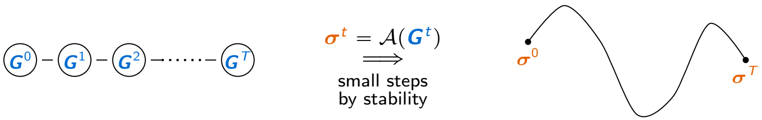

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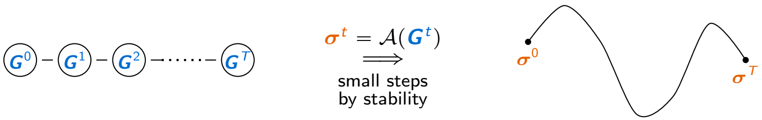

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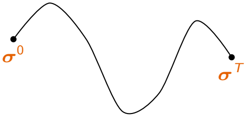

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Chaos property : $\boldsymbol{G}^0$, $\boldsymbol{G}^T$ don't have solutions $\boldsymbol{\sigma}, \boldsymbol{\rho}$ with diff $\leqslant q_{\text{ogp}}$

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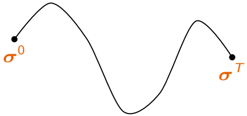

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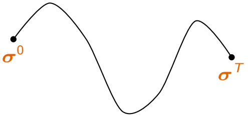

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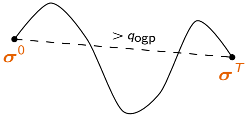

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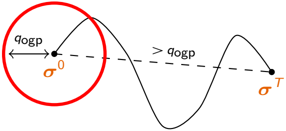

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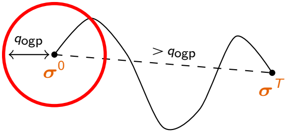

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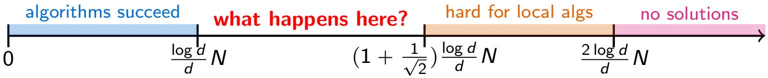
Questions

- Can we show a tighter bound?



- Algorithm classes beyond local algorithms?
- Problems beyond max independent set?
- Finer-grained runtime bounds?

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Instead of $|\text{difference}(\boldsymbol{\sigma}, \boldsymbol{\rho})|$, equivalent to consider **overlap** $(\boldsymbol{\sigma}, \boldsymbol{\rho})/N$, where we encode $\boldsymbol{\sigma}, \boldsymbol{\rho} \in \{0, 1\}^N$

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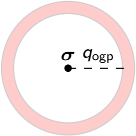

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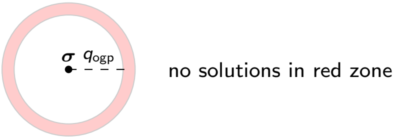

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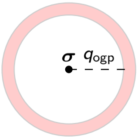

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Known exceptions:

- Random k-XOR-SAT exhibits OGP, but solved by gaussian elimination
- Lattice methods use algebraic structure (Zadik Song Wein Bruna 21)
- Shortest path exhibits OGP but easy (Li Schramm 24)

Outline of talk:

Introduction and motivating problems

Overlap gap property: the basics

Hardness for low degree polynomials

Strong low degree hardness

Other OGPs and SLDH

Further applications and conclusion

Low degree polynomials

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(But see Buhai Hsieh Jain Kothari 25 for counterexample)

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Our main result improves this to $o(1)$

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Theorem (Wein 21)

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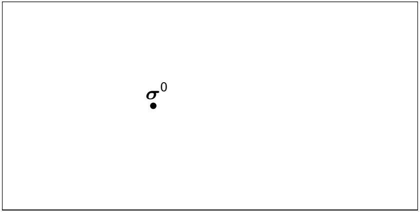

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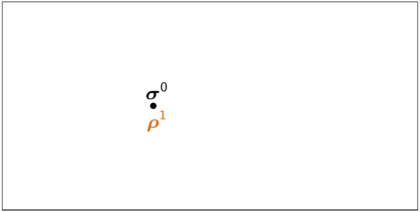

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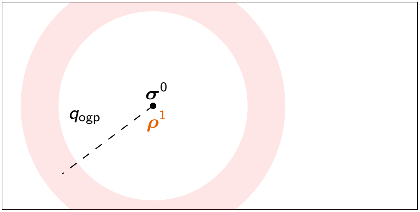

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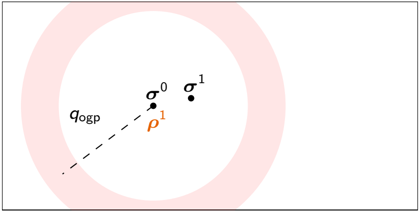

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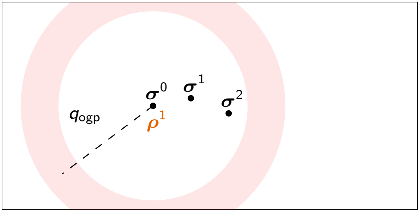

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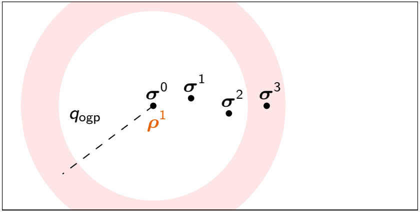

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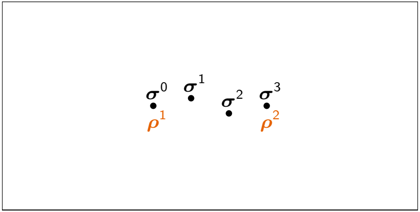

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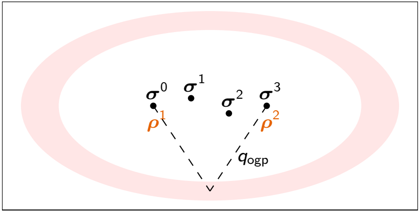

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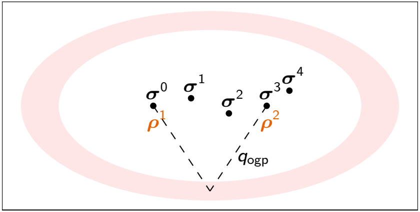

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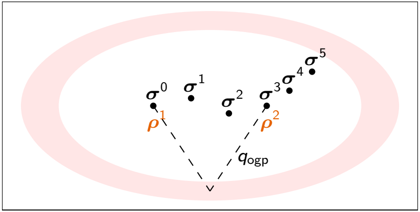

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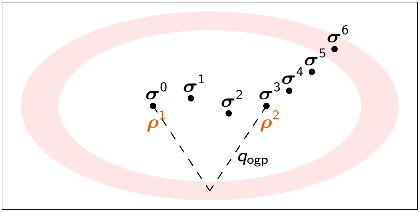

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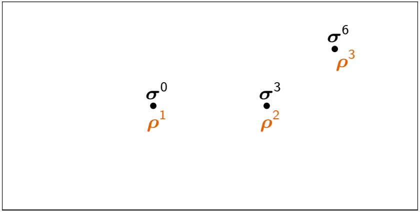

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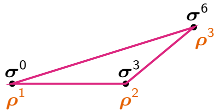

This is the forbidden structure!

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- Random systems of polynomial equations (Montanari Subag 24)
- Largest average submatrix / subtensor (Gamarnik Li 16, Bhamidi Gamarnik Gong 25)

Outline of talk:

Introduction and motivating problems

Overlap gap property: the basics

Hardness for low degree polynomials

Strong low degree hardness

Other OGPs and SLDH

Further applications and conclusion

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Union bound: $p_{\text{solve}} \leqslant 1 - 1/T$ ☹

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This issue quietly plagued the OGP literature for years. Numerous works prove $p_{\text{solve}} \leqslant 1 - e^{-\Omega(\boldsymbol{D})}$ for degree $\boldsymbol{D}$ polynomials:

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We overcome this issue, and actually show $p_{\mathrm{solve}} = o(1)$ for degrees **much larger** than $O(1)$.

Strong low degree hardness

Theorem (H Sellke 25, informal)

If classic/ladder OGP obstruction holds with probability $1 - p_{\text{ogp}}$*, then*

$$\mathbb{P}(\textit{a degree } \boldsymbol{D} = o(\log \tfrac{1}{p_{\text{ogp}}}) \textit{ algorithm succeeds}) = o(1)$$

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This is tight: $\boldsymbol{D} = O(N)$ / time $e^{O(N)}$ can brute force

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- [Symmetric / neg Ising perceptron: $p_{\text{ogp}} = e^{-cN}$, $\boldsymbol{D} = o(N/\log N)$]

Low degree heuristic (Hopkins 18): degree $\boldsymbol{D}$ polys $\approx e^{\boldsymbol{D}}$ time algs
Suggests e.g. $e^{o(N)}$ time algorithms don't beat ALG in max ind set

This is tight: $\boldsymbol{D} = O(N)$ / time $e^{O(N)}$ can brute force
(and in max-clique: $\boldsymbol{D} = O(\log^2 N)$ / time $e^{O(\log^2 N)}$ can brute force)

Strong low degree hardness: proof ideas

Let's revisit **ladder** OGP: consider Markovian sequence of problems

$$\boldsymbol{G}^0 \to \boldsymbol{G}^1 \to \cdots \to \boldsymbol{G}^T$$

$\boldsymbol{G}^{i+1}$ resample **each** edge of $\boldsymbol{G}^i$ with pr ε.

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Strong low degree hardness: initial attempt

We can iterate this dyadically!

$$\mathbb{P}(\mathsf{Stab}(0,\ldots,T)) = \mathbb{E}[\mathbb{P}(\mathsf{Stab}(0,\ldots,\tfrac{T}{2}) \cap \mathsf{Stab}(\tfrac{T}{2},\ldots,T) | \boldsymbol{G}^{\frac{T}{2}})]$$

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- Same proof: $\mathbb{P}(\mathcal{A} \text{ solve all } \boldsymbol{G}^0,\ldots,\boldsymbol{G}^T) \geqslant \mathbb{P}(\mathcal{A} \text{ solve } \boldsymbol{G}^0)^T \equiv p_{\mathrm{solve}}^T$

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- $\mathbb{P}(\nexists \text{ forbidden structure}) = 1 - e^{-cN}$

Strong low degree hardness: initial attempt

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(iterate) $$\geqslant \mathbb{P}(\mathrm{Stab}(0,1))^T \equiv p_{\mathrm{stable}}^T$$

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don't occur simultaneously $\Rightarrow$ $p_{\mathrm{solve}}^T + p_{\mathrm{stable}}^T \leqslant 1 + e^{-cN}$

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Doesn't yet imply $p_{\mathrm{solve}} = o(1)$ ☹

Strong low degree hardness via dyadic Jensen

💡 do dyadic Jensen on merged event Solve&Stab$(0, \ldots, T)$:

$$\Big\{\mathcal{A} \text{ solves } \boldsymbol{G}^0, \ldots \boldsymbol{G}^T \text{ and } \|\mathcal{A}(\boldsymbol{G}^i) - \mathcal{A}(\boldsymbol{G}^{i+1})\| \text{ small for } 0 \leqslant i \leqslant T-1\Big\}$$

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Since $\mathbb{P}(\nexists \text{ forbidden structure}) = 1 - e^{-cN}$ & don't occur simultaneously:

$$\mathbb{P}(\text{Solve\&Stab}(0,1)) \leqslant e^{-cN/T} \qquad \text{is small}$$

Strong low degree hardness via dyadic Jensen

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(more generally, $\boldsymbol{D} = o(\log \frac{1}{p_{\text{ogp}}})$ if $\mathbb{P}(\nexists \text{ forbidden structure}) = 1 - p_{\text{ogp}}$)

Outline of talk:

Introduction and motivating problems

Overlap gap property: the basics

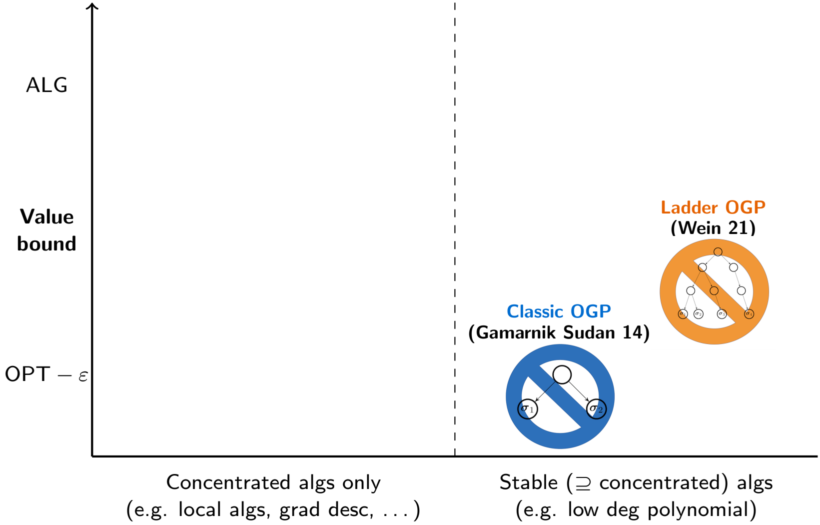
Hardness for low degree polynomials

Strong low degree hardness

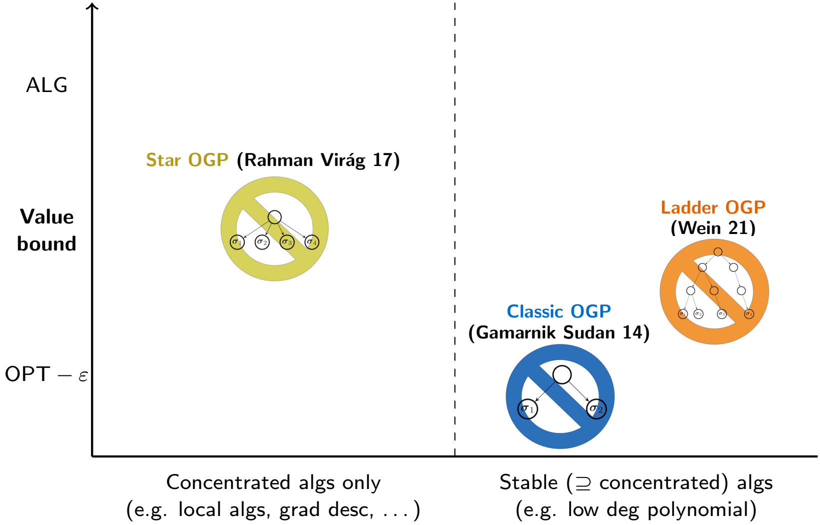
Other OGPs and SLDH

Further applications and conclusion

Comparison of OGPs in general problems



Comparison of OGPs in general problems



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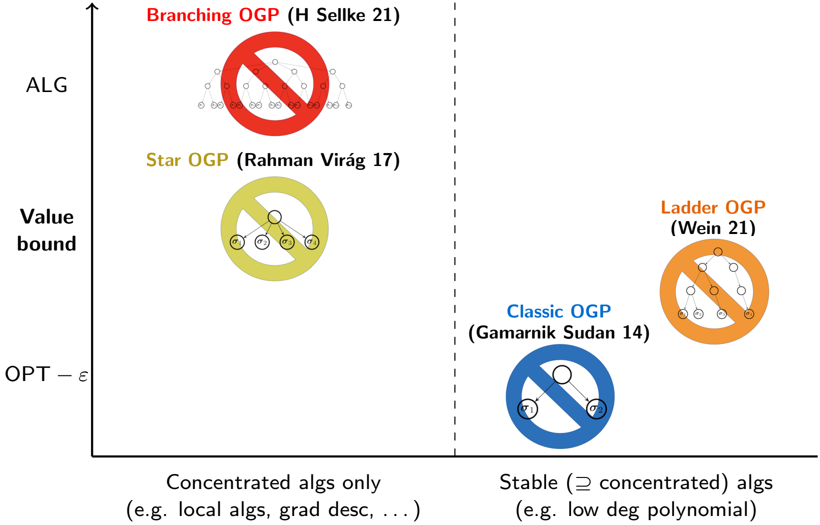

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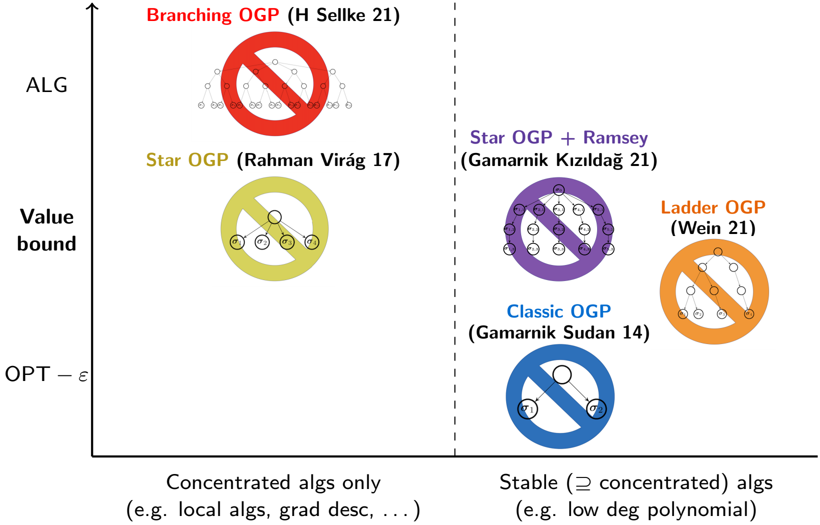

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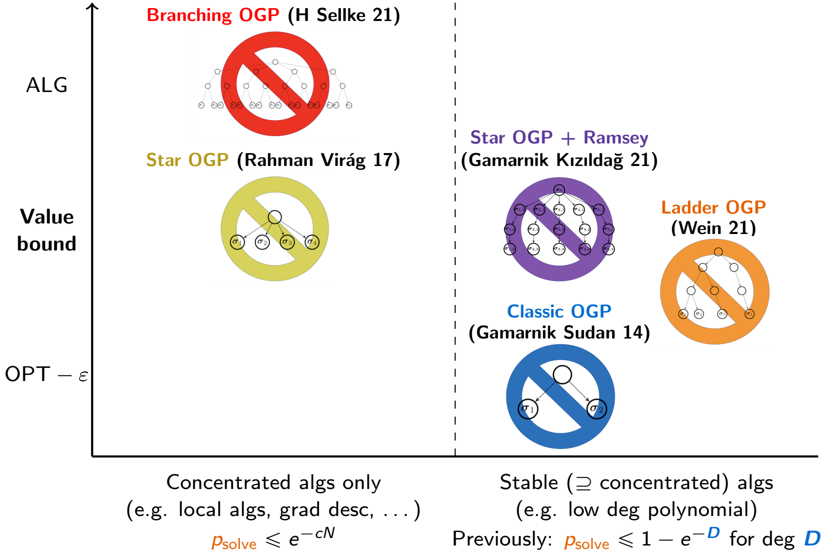

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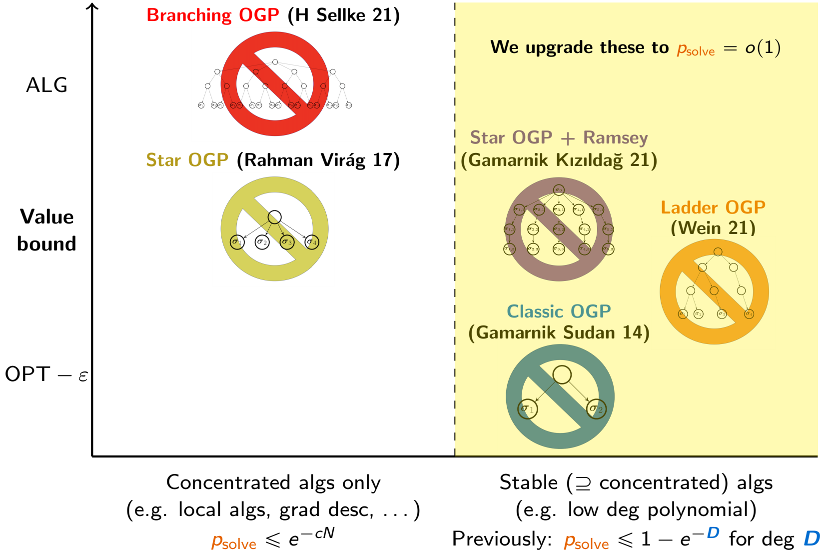

Strong low degree hardness from star OGP + Ramsey

Theorem (H Sellke 25)

If a star OGP holds with probability $1 - p_{\mathrm{ogp}}$, *then*

$$\mathbb{P}\Big(\textit{a degree } \boldsymbol{D} = o(\log \tfrac{1}{p_{\mathrm{ogp}}} / \log\log \tfrac{1}{p_{\mathrm{ogp}}}) \textit{ algorithm succeeds}\Big) = o(1)$$

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Forbidden structure: m solutions with pairwise overlap q_{ogp}.

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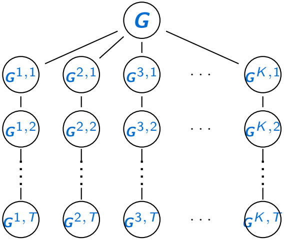
Construct $K \gg 1$ independent resample paths



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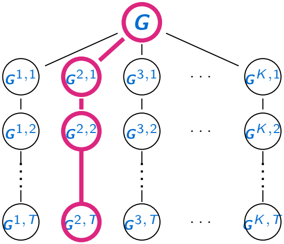

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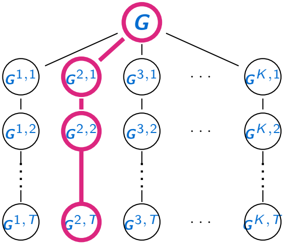

$\text{Solve\&Stab}^{(k)} = \{$on k-th arm, $\mathcal{A}$ solves all & all steps stable$\}$

As before: $\mathbb{P}(\text{Solve\&Stab}^{(k)}) \geqslant \mathbb{P}(\text{Solve\&Stab(one step)})^T$

Strong low degree hardness from star OGP + Ramsey

Theorem (H Sellke 25)

If a star OGP holds with probability $1 - p_{\mathrm{ogp}}$, *then*

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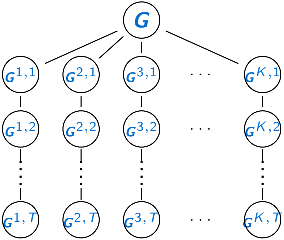
Forbidden structure: m solutions with pairwise overlap q_{ogp}.

Construct $K \gg 1$ independent resample paths



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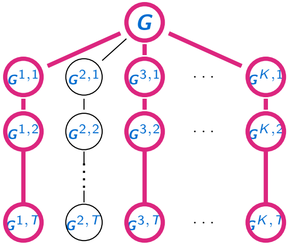

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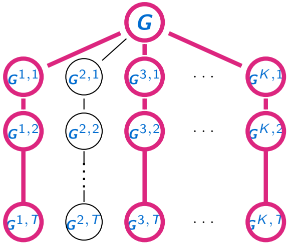

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Ramsey argument of Gamarnik Kızıldağ 21 extracts forbidden structure.

Outline of talk:

Introduction and motivating problems

Overlap gap property: the basics

Hardness for low degree polynomials

Strong low degree hardness

Other OGPs and SLDH

Further applications and conclusion

SLDH for number partitioning problem

NPP: given $g_1, \dots, g_N \overset{IID}{\sim} \mathcal{N}(0,1)$, find $\boldsymbol{\sigma} \in \{\pm 1\}^N$ minimizing

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This is sharp **for all** $1 \ll \boldsymbol{D} \ll N$: deg $\boldsymbol{D}$ achieves $2^{-\tilde{\Omega}(\boldsymbol{D})}$ by brute force.

Relation to shortest path OGP

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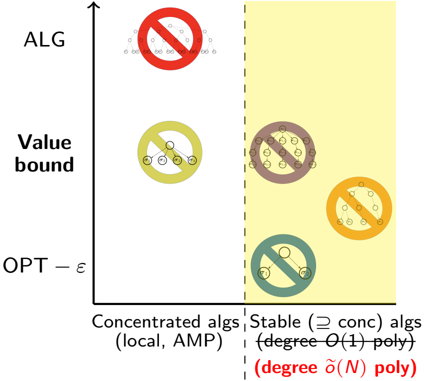
Possible reconciliation: $p_{\text{ogp}} = N^{-\omega(1)}$ necessary for "genuine" hardness

Conclusion

We upgrade existing low deg hardness in random optimization problems: degree $\widetilde{o}(\log \frac{1}{p_{\text{ogp}}})$ algorithms succeed with probability $o(1)$.



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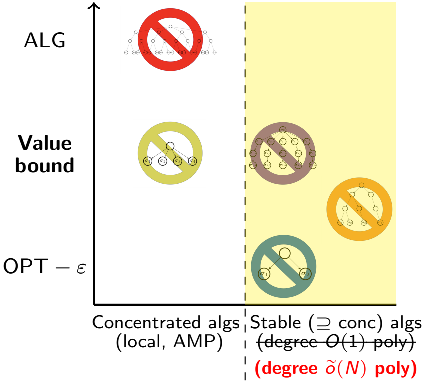

Outstanding challenges:

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- hardness of finding isolated solutions
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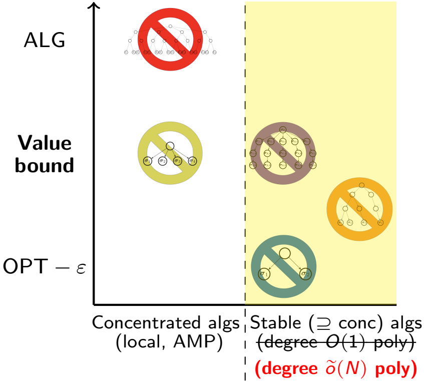

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Thank you!